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ON THE QUESTION OF CALCULATION OF RADIATION IN ONE-DIMENSIONAL MOVEMENT OF MONATOMIC GAS

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ON THE QUESTION OF CALCULATION OF RADIATION IN ONE-DIMENSIONAL MOVEMENT OF MONATOMIC GAS[1]

The development of investigations in various fields of science brings forth ever more frequently hydrodynamic problems for the solution of which consideration has to be given to those physical properties of liquid or gas which have been completely disregarded in classical hydrodynamics since in ordinary problems their action was either not at all manifested or their effect could be disregarded to one degree or another for simplification of calculations. To such properties is related the ability of a material medium in any state to emit and absorb energy. Under certain conditions, which are dealt with in astrophysical problems, dynamic meteorology, and also in several other tasks, radiation so strongly influences results that it becomes impossible to ignore it. The necessity for calculation of radiation in hydrodynamic problems is nearly always linked to high temperatures in some fields of a moving medium. Calculation of radiational transfer of heat, and with very high temperatures and mechanical action of radiation, is linked with great difficulties of both a mathematical and purely physical character (because of the complexity and in many regards vagueness of those physical processes which occur at very high temperatures and pressures).

1. The work was presented at a seminar under the direction of L. I. Sedov at Moscow State University in 1947 - 1948 and at a seminar at the Institute of Mechanics, Academy of Sciences, USSR, in 1949 (for homogeneous gas); sections 5 and 6 (for ionized gas) - at the first All-Union Conference on Aerohydrodynamics at the Institute of Mechanics, Academy of Sciences, USSR, in 1952.

In the present article, an attempt is made in the instance of established one-dimensional movement of monatomic ideal (in a hydrodynamic sense) gas to examine the effect on the characteristic of a stream of radiation and ionization, wherein it is assumed that there is local thermodynamic equilibrium. As an example is considered the problem of structure of a compression wave, for which calculations are made.

Consideration of basic equations of hydrodynamics with calculation of influx of heat due to radiation (in conformity with dynamic meteorology) is contained in an article by Ye. S. Kuznetsov [1]. In the present work is introduced only supplementary information which is linked either with unsteadiness of the radiation field or with mechanical action of the radiation (which were not considered in [1]).

Section 1. Interaction of Radiation with a Material Medium

We will consider that each particle of gas absorbs, radiates, and disperses radiation energy so that an entire stream of gas is permeated with streams of radiation energy. Conversion of other forms of energy into radiation, and, on the contrary, conversion of radiation energy into other forms (for example, into heat) represents a complicated intra-atomic and intra-molecular process and is the subject of special subdivisions of physics. For our purposes, it is desirable in calculation of radiation to as far as possible not disturb the idea of material continuum underlying hydromechanics, and to divert ourselves from consideration of elementary radiators as in hydromechanics we divert ourselves from discretion of the atomic and molecular structure of fluid. Such macroscopic study of a radiation field from the point of view of geometric optics is widely conducted in several branches of physics and makes it possible to quantitatively characterize the interaction of radiation with material continuum (see, for example, [1], [2], [3], [4]). We will introduce definitions and correlations here related, not going into detailed consideration of them.

Intensity of Radiation. The quantity of radiation energy dE_ν which is transferred by an elementary ground $d\sigma$ in a given direction s (Figure 1) (i.e., by "rays" contained in an infinitesimal solid angle $d\Omega$ circumscribed around the direction s), for a short time dt, of spectral composition, and contained in an interval of frequencies from ν to $\nu + d\nu$, may be expressed by the formula:

$$dE_\nu = I_\nu \, d\nu \, d\sigma \cos(n.\ s) \, d\Omega \, dt \tag{1}$$

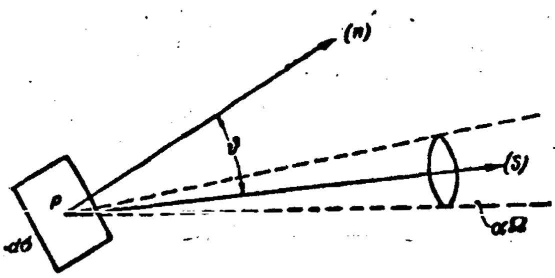

Fig. 1.

The coefficient of proportionality I_ν depends on the frequency of radiation ν, direction s, time t, and the coordinates of point P, around which is constructed ground $d\sigma$, and is called the specific intensity of radiation (or else it is simply called the intensity of radiation, with the brightness of radiation the strength of radiation). If we sum up the intensities of all frequencies in the given point P, in a given moment of time t, and for radiation of a given direction s, then we obtain the integral intensity

$$I = \int_0^\infty I_\nu \, d\nu \tag{2}$$

Stream of Radiation. The magnitude

$$H_{\nu n} = \int_{4\pi} I_\nu \cos(n,\ s)\, d\Omega \tag{3}$$

represents the complete quantity of radiation energy of the frequency ν of all directions, passing in a unit of time through a unit of area, the orientation of which in space is given with the normal n (integration in all directions is designated by 4π, provided with the sign of the integral). This magnitude is a new characteristic of the radiation field, not depending on the direction of the ray.

$H_{\nu n}$ may be considered as a projection to the normal n of a vector $\vec{H}_\nu$. The magnitude $\vec{H}_\nu$ and also the magnitude $H_{\nu n}$ are called the stream of radiation energy of frequency ν. If we integrate it with all frequencies, we obtain an integral stream of radiation energy not dependent on frequency:

$$H_\nu = \int_{4\pi} I \cos(n.\ s)\, d\Omega. \tag{4}$$

Coefficient of Radiation. Emission (radiation) of radiation energy is an atomic-molecular process. In order that the macroscopic examination made by us correspond to actuality, it is necessary that the elementary volume of gas selected be large enough that the total energy radiated by it be completely determined by macroscopic characteristics of the gas filling this volume, i.e., by the temperature, pressure, and density; at the same time, this volume should also be small enough that within this volume these characteristics can be considered constant (with precision to the small of a higher order) and determined by the state of the gas.

If a material medium in a chosen element of a volume in the limits of the solid angle $d\Omega$ radiates in the time dt a quantity of energy equal to dE_ν, whereto frequencies of radiation are changed to a narrow interval from ν to $\nu + d\nu$, then it may be supposed that:

$$dE_\nu = \eta_\nu \, d\nu \, d\Omega \, dt \, \rho \, d\tau, \tag{5}$$

where η_ν is the mass coefficient of radiation. For natural radiation in an isotrope body η_ν is considered to be not dependent on direction but dependent on frequency ν, the coordinates of point P, and on time t.

Coefficient of Absorption. Weakening of the intensity of each ray when it passes through a material medium, on a sufficiently small segment of a path, may be considered proportional to the length of the path ds and to the intensity of radiation I_ν, and therefore, the magnitude by which the intensity decreases may be written:

$$\rho \alpha_\nu I_\nu \, ds, \tag{6}$$

where α_ν is the mass coefficient of absorption. It is dependent on frequency ν, coordinates of point P, and on time t.

With absorption, radiation energy is transformed into thermal movement of elementary particles.

Coefficient of Dispersion. Pure dispersion is linked with redistribution of the intensity of radiation by directions without change of frequency. Decrease of intensity due to dispersion on a small segment of a path of a ray passing through a material medium equals:

$$\rho \sigma_\nu I_\nu \, ds. \tag{7}$$

Here, α_ν is the mass coefficient of dispersion. It is dependent on frequency ν, coordinates of the point, and on time.

Indicatrix of Dispersion. The distribution of intensity of dispersed radiation (7) with direction s in all possible directions s' in any point in a moment of time t may be described by the function:

$$\frac{1}{4\pi} \gamma_\nu (P, t; s, s'), \tag{8}$$

bearing the name indicatrix of dispersion, function of dispersion, or law of dispersion. With the aid of this function, the expression

$$\rho \sigma_\nu I_\nu ds \cdot \frac{1}{4\pi} \gamma_\nu (P, t; s, s') d\Omega' \tag{9}$$

may be treated as that part of all dispersed energy which by the act of dispersion is deflected inside of the solid angle $d\Omega'$ in direction s'. This function, apparently, satisfies the condition of normalization

$$\frac{1}{4\pi} \int_{4\pi} \gamma_\nu (P, t; s, s') d\Omega' = 1. \tag{10}$$

The speed of diffusion of radiation energy will be considered equal to the speed of light c in a vacuum. In the time dt, the energy passing through the ground $d\sigma$ in a given direction s fills a volume

$$d\tau = d\sigma \cos(n, s) c \cdot dt.$$

The density of energy will then be equal to:

$$\epsilon'_{R\nu} \cdot d\nu = \frac{dE_\nu}{d\tau} = \frac{1}{c} I_\nu d\nu d\Omega.$$

If we consider rays of a given spectral composition, going in all possible directions, having integrated the last expression in all directions, we will then have the density of energy of frequency ν in a given point P in a moment of time t in such a form:

$$\epsilon_{R\nu} d\nu = \frac{1}{c} \int_{4\pi} I_\nu d\nu d\Omega \tag{11}$$

Summing up this expression in all frequencies, we obtain the integral density of radiation energy

$$\epsilon_R = \frac{1}{c} \int_{4\pi} I d\Omega. \tag{12}$$

For isotropic radiation we obtain:

$$\epsilon_{R\nu} = \frac{4\pi}{c} I_\nu, \quad \epsilon_R = \frac{4\pi}{c} I. \tag{13}$$

Each photon carries energy $h\nu$ (h - constant of Plank, ν - frequency) and impulse $h\nu/c$, so that with the transfer through ground $d\sigma$

of energy (1) is carried also the impulse

$$\frac{1}{c} I_\nu \, d\nu \, d\sigma \cos(n, s) \, d\Omega \, dt$$

in the direction of diffusion (direction s). Compilation of this impulse along direction s' equals:

$$\frac{1}{c} I_\nu \, d\nu \, d\sigma \cos(n, s) \cos(s', s) \, d\Omega \, dt.$$

Integrating in all directions s, we obtain:

$$\frac{1}{c} d\sigma \, dt \int_{4\pi} I_\nu \, d\nu \cos(n, s) \cos(s', s) \, d\Omega. \tag{14}$$

The tension at point P, arising due to radiation transfer, is the speed of transfer of the impulse, related to a unit of area, through an infinitesimal ground constructed around point P.

This magnitude depends on the direction of the normal and may be written as

$$\vec{\underline{p}}_{R\nu n} \, d\nu = \vec{\underline{n}}^{\circ} ((p_{R\nu ij})) \, d\nu, \tag{15}$$

where $((P_{R\nu ij}))$ is the tensor of radiation tensions (orthogonal symmetrical tensor of the second rank), and $\vec{\underline{n}}^{o}$ is the unit vector directed along the normal. Compilations of the tensor $((P_{R\nu ij}))$ are obtained in the Cartesian system of coordinates from (14) in the following form:

$$p_{R\nu ij} = -\frac{1}{c} \int_{4\pi} I_\nu \cos(s, i) \cos(s, j) \, d\Omega, \tag{16}$$

where i and j = x, y, and z. The minus sign corresponds to the compression of the selected volume by the outer radiation field.

For the entire spectrum we have:

$$p_{Rij} = -\frac{1}{c} \int_{4\pi} I \cos(s, i) \cos(s, j) \, d\Omega, \tag{17}$$

where

$$p_{Rij} = \int_0^\infty p_{R\nu ij} \, d\nu. \tag{18}$$

For isotropic radiation

$$p_{R\nu ij}=0, \quad i\neq j; \tag{19}$$

$$p_{R\nu ii}=-\frac{4\pi}{3c}I_\nu=-\frac{\varepsilon_{R\nu}}{3}. \tag{20}$$

The sum of diagonal components of the tensor $((P_{R\nu ij}))$ equals, apparently,

$$\sum p_{R\nu ii}=-\varepsilon_{R\nu}. \tag{21}$$

If we introduce "hydrostatic" pressure

$$p_{R\nu}=-\frac{1}{3}\sum p_{R\nu ii}, \tag{22}$$

we obtain:

$$p_{R\nu}=\frac{\varepsilon_{R\nu}}{3}, \tag{23}$$

and also

$$p_R=\frac{\varepsilon_R}{3}. \tag{24}$$

For isotropic radiation

$$p_{R\nu ii}=-p_{R\nu}; \quad p_{Rii}=-p_R. \tag{25}$$

The introduced concepts and correlations are sufficient to describe a radiation field of a moving medium and to compile a system of equations of hydromechanics with calculation of radiation.

An equation of transfer of radiation describes the change of intensity of radiation with diffusion of a ray of frequency ν in any direction s. We will select in this direction two points P and P', being separated one from the other by a distance ds. We will construct around the ray s a right cylinder with bases passing through points P and P'. In the time δt through the ground containing point P, inside of the cylinder in the direction s inside of the solid angle $d\Omega$ passes the following quantity of radiation energy (element of energy):

$$\delta E_\nu=I_\nu(P,\ t;\ s)\,d\nu\, d\Omega\, d\sigma\, \delta t.$$

This element with a speed of light c will move inside of the cylinder and in the interval of time $dt = \frac{ds}{c}$ will go out through the other base of the cylinder; whereto, the quantity of energy inside of this element is changed as a result of radiation, absorption, and dispersion by matter along the ray s and will equal upon exit from the cylinder:

$$\delta' E_\nu = I_\nu (P', t + dt; s)\, d\nu\, d\Omega\, d\sigma\, \delta t.$$

Increase of energy

$$\delta'' E_\nu = \left(\frac{\partial I_\nu (P, t; s)}{\partial s} ds + \frac{\partial I_\nu (P, t; s)}{\partial t} dt \right) d\nu\, d\Omega\, d\sigma\, \delta t,$$

with precision to an infinitesimal magnitude of a higher order is made up of radiation in the given direction s:

$$\rho \eta_\nu\, d\nu\, ds\, d\sigma\, d\Omega\, \delta t,$$

of weakening as a result of absorption and dispersion of energy from a given direction to all sides inside of the cylinder:

$$\rho (\alpha_\nu + \sigma_\nu) I_\nu\, d\nu\, ds\, d\sigma\, d\Omega\, \delta t,$$

of increase due to dispersion in accordance with formula (8) from other directions s' in the given direction s:

$$\frac{1}{4\pi} \rho \sigma_\nu\, d\nu\, ds\, d\sigma \int_{4\pi} I_\nu (P, t; s') \gamma_\nu (P, t; s, s')\, d\Omega'\, d\Omega\, \delta t.$$

Combining all these additions together and equating their $\delta''E$, we obtain after cancellation by $d\nu\, \rho\, ds\, d\sigma\, d\Omega\, \delta t$ the sought equation:

$$\frac{1}{\rho} \frac{\partial I_\nu}{\partial s} + \frac{1}{\rho c} \frac{\partial I_\nu}{\partial t} = \eta_\nu + \frac{\sigma_\nu}{4\pi} \int_{4\pi} I_\nu (P, t; s') \gamma_\nu (P, t; s, s')\, d\Omega' - (\alpha_\nu + \sigma_\nu) I_\nu. \tag{26}$$

For a stationary radiation field, the second member in the left part of the equation disappears. In some cases it may be discarded, considering it small in comparison with other members, because of the smallness of the multiplier $\frac{1}{c}$. If pure dispersion is not considered ($\alpha_\nu = 0$), then for a stationary instance equation (26) assumes the form

$$\frac{1}{\rho} \frac{\partial I_\nu}{\partial s} = \eta_\nu - \alpha_\nu I_\nu. \tag{27}$$

In this same form may also be written the equation of transfer of radiation with calculation of dispersion, but then it is necessary to somewhat differently determine coefficients α_ν and η_ν, whereto η_ν, in general, will depend on direction (see Section 3).

Influx of Heat Due to Radiation. We will choose a surface Σ with a volume of matter τ. The influx of heat due to radiation in the element of mass $\rho d\tau$ during time dt will be:

$$\rho q_R d\tau.$$

Having summed up this expression in accordance with volume τ and having equated obtained differences of increase of heat in volume τ due to absorption and diminutions due to radiation (dispersion of components will not be given), we obtain the expression for specific influx of heat q_R in the form

$$q_R = \int_0^\infty \alpha_\nu d\nu \int_{4\pi} I_\nu d\Omega - 4\pi\eta; \quad \eta = \int_0^\infty \eta_\nu d\nu. \tag{28}$$

The quantity of radiation energy which will be introduced in a unit of time inside a surface Σ in all frequencies, may be determined by summation of expression (1) for all frequencies, directions, and for the entire surface. We obtain:

$$-\int_\Sigma H_n d\sigma = -\int_\tau \operatorname{div} \vec{H} d\tau.$$

Going on to the limit with $\tau \to 0$, we find the magnitude of influx of energy due to radiation as related to the unit of time and volume:

$$\rho Q_R = -\operatorname{div} \vec{H}. \tag{28'}$$

We multiply (26) by $d\Omega d\nu$ and integrate in all directions and all frequencies, and we have

$$\frac{1}{\rho} \operatorname{div} \vec{H} + \frac{1}{\rho} \frac{\partial \varepsilon_R}{\partial t} = 4\pi\eta - \int_0^\infty \alpha_\nu d\nu \int_{4\pi} I_\nu d\Omega. \tag{29}$$

Comparing this with (28), we obtain another expression for magnitude of specific influx of heat:

$$q_R = -\frac{1}{\rho}\operatorname{div}\vec{H} - \frac{1}{\rho}\frac{\partial \epsilon_R}{\partial t}. \tag{30}$$

Utilizing this equation jointly with (28'), we arrive at the correlation:

$$\rho Q_R = \rho q_R + \frac{\partial \epsilon_R}{\partial t},$$

i.e., the radiation energy flowing into the elementary volume of the medium goes partly to heating and partly to increase of density of radiation energy in this volume.

For isotropic radiation, $\vec{H} = 0$, and therefore

$$Q_R = 0; \quad q_R = -\frac{1}{\rho}\frac{\partial \epsilon_R}{\partial t}.$$

If the radiation field is moreover also stationary, then

$$q_R = 0.$$

Section 2. Common Equations of Movement of a Material Medium with Calculation of Radiation

A system of equations describing movement of a solid medium, namely: an equation of transfer of a quantity of movement, an equation of transfer of mass, and an equation of transfer of energy, is worked out on very common assumptions and allows various processes inside the medium. Specifically, it is used for description of movement of a solid medium with calculation of radiation, i.e., in the radiation field, if only the magnitudes which enter into it are determined in a corresponding manner.

1. An equation of movement of a solid medium in tensions with calculation of radiation is not distinguished in form from an equation of movement without calculation of radiation:

$$\frac{1}{\rho}\frac{d\vec{v}}{dt} = \rho\vec{F} + \operatorname{div}((P_{ij})). \tag{1}$$

with this there are only the differences that with the tensor of tensions $((P_{ij}))$ should here be understood the sum of two tensors: the tensor $((P_{ij}))$ of tensions of a solid medium and the tensor of radiation tensions $(P_{Rij}))$:

$$((P_{ij})) = ((p_{ij})) + ((p_{Rij})). \tag{2}$$

Other symbols have their own usual meaning: ρ - density, $\vec{\nu}$ - vector of speed, and its projections u_x, u_y, u_z : $\vec{F}$ - vector of mass strength; i and j = x, y, z. Ordinarily, the tensor of tensions in a viscous fluid is presented in the form of the sum of the tensor of hydrostatic pressure and the tensor of viscous tensions:

$$((p_{ij})) = -p((\delta_{ij})) + ((\tau_{ij})), \tag{3}$$

where $((\delta_{ij}))$ is the single tensor, so that

$$p_{ij} = -p + \tau_{ij}, \quad \text{if } i \neq j.$$
$$p_{ii} = \tau_{ii}.$$

In precisely this same way we may also divide the tensor of radiation tensions into two parts:

$$((p_{Rij})) = -p_R((\delta_{ij})) + ((\tau_{R_{ii}}));$$
$$p_{Rij} = \begin{cases} -p_R + \tau_{R_{ij}}, & i \neq j \\ \tau_{R_{ij}}, & i = j \end{cases} \tag{4}$$

the first tensor corresponds to the isotropic radiation field, and the second to oblique radiation tensions if we assume [4]:

$$I_\nu = \bar{I}_\nu + \bar{\bar{I}}_\nu, \tag{5}$$

where $\bar{I}_\nu$ is defined as the magnitude of "output":

$$\bar{I}_\nu = \frac{1}{4\pi}\int_{4\pi} I_\nu \, d\Omega = 4\pi c a_{R\nu}. \tag{6}$$

Substituting this in (12, section 1), we have,

$$\tau_{Rij} = -\frac{1}{c}\int_{4\pi} \bar{\bar{I}}_\nu \cos(i, s)\cos(j, s)\, d\Omega. \tag{7}$$

If we multiply the equation of transfer of radiation by the cosine (i, n) dνd and integrate with all frequencies and directions, then we obtain:

$$\operatorname{div}_i((p_{Rij})) = -\frac{1}{c^2}\frac{\partial H_i}{\partial t} - \frac{\rho}{c}\int_0^\infty (\alpha_\nu + \sigma_\nu) H_{\nu i}\, d\nu -$$

$$-\frac{1}{4\pi}\frac{\rho}{c}\int_0^\infty \sigma_\nu\, d\nu \int_{4\pi} \cos(s, l)\, d\Omega \int_{4\pi} I_\nu(P, t; s')\gamma_\nu(P, t; s, s')\, d\Omega'. \tag{8}$$

For isotropic radiation

$$\operatorname{div}((p_{Rij})) = 0. \tag{9}$$

For a viscous fluid, components of a tensor of viscous tensions $((\tau_{ij}))$ are expressed in hydromechanics by components of a tensor of deformation in the following form:

$$\tau_{ij} = \begin{cases} -p + \lambda \operatorname{div} \vec{v} + 2\mu e_{i,i} \\ 2\mu e_{i,j},\ i \neq j \end{cases}, \tag{10}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right),$$

μ, λ - coefficients of viscosity.

2. <u>An equation of conservation of mass</u>, if limited to cases in which in the considered field sources and discharges of mass are absent, also maintains its usual form in calculation of radiation:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{v} = 0. \tag{11}$$

If there are sources or discharges, then in the right part should be supplied the power of the sources (discharges).

3. <u>Equation of energy</u>. We select in the stream of fluid some volume τ, limited by the motionless surface Σ. We apply the law of conservation of energy to it. The expression of kinetic energy $\frac{\rho \vec{v}^2}{2} d\tau$ of a particle of fluid is not changed by the presence of a radiation

field in our statement of the problem. As regards internal energy, by internal energy of a particle $\rho \varepsilon d\tau$ should now be understood the sum of internal energy of considered movements of elementary particles (molecules, atoms, etc.) ε_T and of energy of radiation ε_R, whereto ε_T will be related to the unit of mass and ε_R we previously related to the unit of volume, and therefore:

$$\varepsilon = \varepsilon_T + \frac{\varepsilon_R}{\rho}. \tag{12}$$

To the element of surface $d\sigma$ are added surface forces, tension of the main vector of which $\vec{P}_n$ will consist of the tension of the main vector of surface forces due to internal tensions of the fluid, i.e., due to transfer through the surface of elementary impulses by elementary particles of fluid, and of the tension of the main vector due to radiation tensions - due to transfer of impulses by photons, i.e.,

$$\vec{P}_n = \vec{p}_n + \vec{p}_{Rn}. \tag{13}$$

We will consider the work of mass forces, and also the influx of heat due to thermal conductivity and radiation transfer of energy.

By the usual methods of derivation of an equation of energy (with utilization of an equation of movement and an equation of indissolubility) we obtain:

$$\rho \frac{d\varepsilon}{dt} = \operatorname{div}(k \operatorname{grad} T) + \operatorname{div} \vec{H} = \pi \equiv 0, \tag{14}$$

where

$$\pi = \sum_{i,j} P_{ij} e_{ij}, \tag{15}$$

$\varkappa$ — coefficient of thermal conductivity.

If (3), (4), and (12) are utilized, then (14) is rewritten as:

$$\rho \frac{d\varepsilon_T}{dt} + \frac{d\varepsilon_R}{dt} - \frac{p + \frac{4}{3}\varepsilon_R}{\rho} \frac{d\rho}{dt} - \operatorname{div}(k \operatorname{grad} T) + \operatorname{div} \vec{H} - \Phi = 0;$$

$$\Phi = \sum_{ij} (\tau_{ij} + \tau_{Rij}) l_{ij}.$$

If $\Phi = 0$, vectors of degrees T, $\vec{\upsilon}$, and $\vec{H}$ are parallel, and mass forces are conservative ($\vec{F}$ = degrees U), then for stationary movement may be obtained the first integral - the generalized integral of Bernoulli:

$$\varepsilon_T + \frac{\upsilon^2}{2} - U + \frac{p + \frac{4}{3}\varepsilon_R}{\rho} - \frac{1}{\rho\upsilon}\frac{dT}{ds} + \frac{H}{\rho\upsilon} = \text{const} \tag{17}$$

the right part is constant along the line of current.

The system of equations (1), (11), and (14), if usual hypotheses concerning the relation of a tensor of tensions $((P_{ij}))$ to a tensor of speeds of deformations $((e_{ij}))$ are introduced, in general proves to be open. In addition, it is necessary to enlist a thermodynamic equation of the state of matter. Considering all coefficients which will be figured in the equations (coefficients of viscosity, coefficient of thermal conductivity, etc.) as constant or as known functions of variables already introduced into the equation, we close the system if radiation is not considered. In calculation of the latter, a number of other magnitudes appear, but all of them, as was shown above, may be expressed by intensity of radiation I_ν. Including in the system the equation of transfer of radiation (22, section 2), we introduce a number of new magnitudes: α_ν, η_ν, σ_ν, γ_ν.

Part of them we consider known functions which may not be determined by hydromechanical methods. To them are related α_ν, η_ν, and γ_ν; as regards the coefficient of radiation η_ν, it is an unknown function and for its determination an additional correlation is necessary. In the quality of such a correlation is usually taken Kirchhoff's law.

$$\frac{\eta_\nu}{\alpha_\nu} = B_\nu, \tag{18}$$

where B_ν is the function of radiation of Plank, or in the integral form:

$$\frac{\eta}{\alpha} = B = \frac{\sigma}{\pi} T^4. \tag{19}$$

The latter formulas are justified, strictly speaking, only in a case of thermodynamic equilibrium and may be used in those problems of hydromechanics in which the hypothesis concerning local thermodynamic equilibrium is justified. There may also be obtained other correlations describing the state of radiation which may be used when Kirchhoff's law proves impracticable [1].

Section 3. One-Dimensional Movement of Radiating Gas

For one-dimensional movement, the system of basic equations may be significantly simplified.

We will consider that movement of a material medium in a radiation field occurs along the direction of the system of radiation, i.e., along the gradient of temperature. We will assume that all characteristics of the gas and radiation field depend on only one coordinate x and in the case of non-stationary movement - also on time t. Thus, in each point of any plane perpendicular to the direction of the speed of current, all radiation and hydrodynamic magnitudes will be considered completely identical. Only one magnitude will depend on direction, namely - specific intensity I_ν. Considering the medium, thus, as flat-stratified, we must assume as a valid symmetry also its axis symmetry relative to the radiation field, i.e., consider that the intensity of radiation depends only on one angle ϑ formed by a ray with the direction of speed of movement of the gas, and does not depend on the other two angles (or other two guiding cosines). If we consider only the radiation of the medium itself, considering there is no radiation from outside sources, then for a one-dimensional problem and for a homogeneous medium, the observations made follow as they are. This remains justifiable also in that case in which there is an outer source of radiation but the stream from it is directed along the vector of the speed of movement of the medium.

Non-diagonal components of a tensor of radiation pressure in the case of movement along an axis x become zero:

$$p_{R\nu ij}=0,\ i\neq j.$$

From considerations of symmetry

$$p_{R\nu zz}=p_{R\nu yy}.$$

In addition, we have:

$$H_{\nu y}=H_{\nu z}=0;\ H_y=H_z=0.$$

In an equation of transfer, instead of differentiation by direction of the ray s it is now convenient to introduce differentiation along axis x, having utilized the equation:

$$\frac{\partial}{\partial s}=\cos\theta\frac{\partial}{\partial x}.$$

The basic system of equations is written in this particular case as:

$$\frac{du}{dt}=X+\frac{1}{\rho}\frac{\partial}{\partial x}(p_{xx}+p_{Rxx}); \tag{1}$$

$$\frac{d\rho}{dt}=-\rho\frac{\partial u}{\partial x}; \tag{2}$$

$$\rho\frac{d\varepsilon_T}{dt}+\frac{d\varepsilon_R}{dt}-\frac{\varepsilon_R-p_{xx}-p_{Rxx}}{\rho}\frac{d\rho}{dt}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}-H_x\right); \tag{3}$$

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t}+\cos\theta\frac{\partial I_\nu}{\partial x}=\rho\eta_\nu+\rho\frac{\sigma_\nu}{4\pi}\int\limits_{4\pi}I_\nu(P,t;s')\gamma_\nu(P,t;s,s')d\Omega'-$$
$$-(\alpha_\nu+\sigma_\nu)I_\nu. \tag{4}$$

If the movement is stationary and if mass forces are conservative

$$X=\frac{dU}{dx}, \tag{5}$$

then the system allows the first integrals:

$$\rho u=m, \tag{6}$$

$$m\left(\varepsilon_T+\frac{\varepsilon_R}{\rho}-U+\frac{u^2}{2}-\frac{p_{xx}+p_{Rxx}}{\rho}\right)+H_x-k\frac{dT}{dx}=l. \tag{7}$$

If U = 0, then system (1) - (3) is reduced to the following levels:

$$\rho u=m; \tag{8}$$

$$p_{xx}+p_{Rxx}-mu=n; \tag{9}$$

$$k\frac{dT}{dx}-m\left(\varepsilon_T+\frac{\varepsilon_R}{\rho}\right)-nu+\frac{mu^2}{2}-H_x=l, \tag{10}$$

m, n, l — (constant integrations).

If the radiation field is stationary or if the member $\frac{1}{c}\frac{\partial I_\nu}{\partial t}$ is disregarded, considering it small in comparison with others, then the equation of transfer of radiation (4) for one-dimensional movement is rewritten in the following form:

$$\frac{\cos\vartheta}{\rho\varkappa_\nu}\frac{\partial I_\nu}{\partial x}=J_\nu-I_\nu, \tag{11}$$

where

$$\varkappa_\nu=\alpha_\nu+\sigma_\nu,$$

$$J_\nu=\frac{\eta_\nu}{\varkappa_\nu}+\frac{\sigma_\nu}{4\pi\varkappa_\nu}\int\limits_{4\pi} I_\nu(P,\ t;\ s')\,\gamma_\nu(P,\ t;\ s,\ s')\,d\Omega', \tag{12}$$

J_ν - the function of radiation, determines the quantity of radiation energy emitted by a unit of mass by way of radiation and dispersion; in general, it depends on the coordinate x, direction (i.e., on the angle ϑ), time t, and frequency ν. If the indicatrix of dispersion does not depend on ϑ, for example in the case of isotropic dispersion $\gamma = 1$), or if dispersion is not considered ($\alpha_\nu = 0$), then J_ν does not depend on direction.

In mean form by frequencies, equation (11) is written in the form:

$$\frac{\cos\vartheta}{\rho\varkappa}\frac{\partial I}{\partial x}=J-I, \tag{13}$$

[*Translator's note*: Pages 90 and 91 of original not available].

If the temperature is so high that complete dissociation of molecules has occurred, then the internal energy of the gas will consist of energy of the now monatomic gas ($\gamma = \frac{5}{3}$) and energy of dissociation:

$$\varepsilon_T = \frac{3}{2}\frac{p}{\rho} + \frac{U_d}{\mu},$$

where U_d is energy of dissociation in calculation for one mole.

We will consider monatomic gas in more detail. Let us assume that the temperature was sufficiently high that ionization had vital importance. The medium in this case will be considered a mixture of three perfect gases: gas consisting of neutral atoms, gas of ions, and electronic gas.

Two processes lead to loss of electrons by atoms: thermal ionization produced by collisions of atoms with free electrons and also with atoms and ions, and photoionization, which is produced by light quanta (photons) extracting electrons from atoms. Simultaneously with processes of ionization must also occur processes of recombination of ions and electrons into neutral atoms. Herewith, the surplus of energy either gives itself up to some third particle or is emitted in the form of a light quantum. Photoionization has predominant importance at very low pressures.

At normal and very high pressures, as a problem concerning an intensive compression wave, thermal ionization predominates. But the latter form of ionization we will consider in the future.

Common correlations for ionized gas may be obtained by methods of elementary thermodynamics, considering ionization as a series of chemical reactions, as well as by methods of statistical physics.

We will assume that in every point there is ionizational equilibrium and that each free particle of gas possesses energy $\frac{3}{2}kT$ and impulse kT, as follows from the kinetic theory of gases. In a case of thermodynamic equilibrium, the pressure of gas equals:

$$p = NkT,$$

where N is the number of elementary particles in a unit of volume, and k is the constant of Boltzmann. As long as all particles are independent, the pressure of the gas will consist of the sum of partial pressures of electronic p_e, ionic p_u, and atomic p_a gases.

For determination of pressure it is necessary to calculate the concentration of neutral atoms, ions, and electrons dependent on temperature and density of the gas. Increase of the temperature of gas gives rise to not only increase of the degree of ionization (concentration of electrons), but also, generally speaking, to excitation of deeper electronic levels of neutral and ionized atoms. We will designate the number of electrons in a unit of volume of gas of given conditions by n_e, the number of neutral atoms by n_a, and the number -times ionized atoms by n_r, whereto r, apparently, may in general change from 1 to $\mathcal{R}$, where $\mathcal{R}$ is the ordinal number of the element in the periodic system of Mendeleyev, and the number of ions n_u will equal $\sum_r n_r$. The total number of particles in a unit of volume will equal:

$$N = n_e + n_a + n_u,$$

accordingly, the pressure of such a gas is expressed by the formula:

$$p = (n_e + n_a + n_u)\, kT. \tag{1}$$

Let us assume that conditions are such that only single (one-time) ionization takes place, i.e., only one outer electron or none is torn from the atom. This may be assumed by reason that the energy of the disengagement of the first electron is always significantly less than the energy of the disengagement of the second electron, and appreciable double ionization may occur only after nearly complete single ionization.

Designating the total number of atoms in a unit of volume by N_o, we obtain for single ionization:

$$p=(1+z)N_0kT, \tag{2}$$

where z is the degree of ionization, i.e., the number of ionized atoms relative to the total number of atoms. Multiplying and dividing the right part (2) by the mass of the atom m_a, we have

$$p=\frac{k}{m_a}(1+z)\rho T, \tag{3}$$

or

$$p=\frac{R_0}{\mu_H}(1+z)\rho T, \tag{4}$$

where $\rho = N_o m_a$, i.e., the mass of the electron as compared with the mass of the atom is disregarded, $R_o = \frac{k}{m_H}$ is the universal gas constant, μ_H is the atomic mass of non-ionized gas relative to hydrogen, and m_H is the mass of an atom of hydrogen.

Internal energy resident in a unit of volume will be comprised of kinetic energy of translational movements of elementary particles $\frac{3}{2}kTN$, energy of ionization, and energy of excitation.

For ionization (r = 1), a single time of an ionized atom requires work equal to r-that potential of ionization χ_r, and for ionization of n_r atoms, work required equals $n_r\chi_r$. But to the extent that each of the ions entering into n_r is considered subsequently to be ionized r time, then to these atoms must be ascribed the energy of ionization $n_r\sum_{i=0}^{r}\chi_v$, and all the energy of ionization in calculation for a unit of volume of gas will equal:

$$\sum_r\left(n_r\sum_{i=1}^{r}\chi_i\right). \tag{4a}$$

The energy of excitation in calculation for one particle is represented by the formula known from statistical physics:

$$\varepsilon_r^{(j)} = kT^2 \frac{\partial \ln u_r}{\partial T}, \tag{5}$$

where u_r — sum by states

$$u_r = \sum_s g_{r,s} e^{-\chi_{r,s}/kT}, \tag{6}$$

$\chi_{r,s}$ is the potential of excitation of a particle of the type r in the given case of a r time ionized atom being in a quantum state s (ascribing to the index r the value 0, we will consider neutral atoms), and $g_{r,s}$ is the statistical weight. Summation is conducted by all quantum states of the particle. The energy of excitation of all atoms in a unit of volume will equal:

$$\varepsilon^{(j)} = kT^2 \sum_{r=0}^{\mathfrak{N}} n_r \frac{\partial \ln u_r}{\partial T}. \tag{7}$$

Adding up (4), (5), and (7), we obtain the volumetrical density of the complete internal energy of the gas:

$$\varepsilon'_T = \frac{3}{2} NkT + \sum_{r=1}^{\mathfrak{N}} n_r \sum_{i=1}^{r} \chi_i + kT^2 \sum_{r=0}^{\mathfrak{N}} n_r \frac{\partial \ln u_r}{\partial T}. \tag{8}$$

For once ionized gas we obtain:

$$\varepsilon'_T = \frac{3}{2} N_0 kT(1+z) + N_0 z\chi_1 + N_0 kT^2 \left\{(1-z)\frac{\partial \ln u_0}{\partial T} + \frac{\partial \ln u_1}{\partial T}\right\}, \tag{9}$$

which may also be written thus if the right part is multiplied and divided by the mass of a neutral atom of the considered gas and it is considered that $N_o m_a = \rho$:

$$\varepsilon'_T = \frac{3}{2} \rho \frac{R_0}{\mu_H} T(1+z) + \rho z \frac{\chi_1}{m_a} + \rho \frac{R_0}{\mu_H} T^2 \left\{(1-z)\frac{\partial \ln u_0}{\partial T} + z\frac{\partial \ln u_1}{\partial T}\right\}. \tag{10}$$

The mass density of internal energy will be:

$$\varepsilon_T = \frac{3}{2} \frac{R_0}{\mu_H} (1+z) T + z\frac{\chi_1}{m_a} + \frac{R_0}{\mu_H} T^2 \left\{(1-z)\frac{\partial \ln u_0}{\partial T} + z\frac{\partial \ln u_1}{\partial T}\right\}. \tag{11}$$

The potential of ionization χ_1 is a magnitude which is constant for a given gas and is determined experimentally.

The entropy of a unit of volume of ionized gas is obtained in this form by methods of statistical physics:

$$S=\frac{5}{2}kN+\sum_{r=0}^{\mathfrak{R}}kn_r\left[\ln\frac{(2\pi m_a kT)^{\frac{3}{2}}u_r}{n_r h^3}+7\frac{\partial\ln u_r}{\partial T}\right]+kn_e\ln\frac{(2\pi m_e kT)^{\frac{3}{2}}2}{n_e h^3}. \tag{12}$$

In the last item, instead of a sum by states was introduced the statistical weight of an electron equal to two.

We will also write an expression for the thermal function W and free energy F. Utilizing the known thermodynamic equations:

$$W=\varepsilon_T+pv;$$
$$F=\varepsilon_T-TS,$$

we have:

$$W=\frac{5}{2}NkT+\sum_{r=1}^{\mathfrak{R}}n_r\sum_{i=1}^{r}\chi_i+kT^2\sum_{r=0}^{\mathfrak{R}}n_r\frac{\partial\ln u_r}{\partial T}. \tag{13}$$

$$F=-NkT+\sum_{r=1}^{\mathfrak{R}}n_r\sum_{i=1}^{r}\chi_1-$$
$$-\sum_{r=0}^{\mathfrak{R}}\left[kn_r\ln\frac{(2\pi m_a kT)^{\frac{3}{2}}u_r}{n_r h^3}\right]-kn_e\ln\frac{(2\pi m_e kT)^{\frac{3}{2}}2}{n_e n^3}. \tag{14}$$

In a state of equilibrium, free energy should attain a minimum of dependence on parameters determining the degree of ionization, whereby

$$\frac{\partial F}{\partial n_r}=0,$$

or, differentiating (10) and taking into consideration the condition of the electrical neutrality of the gas as a whole:

$$n_e=\sum_{r=1}^{\mathfrak{R}}rn_r,$$

we obtain an equation determining the concentration of ions:

$$n_r n_e^r=u_r\frac{(2\pi m_a kT)^{3/2}}{h^3}\left\{\frac{2(2\pi m_e kT)^{3/2}}{h^3}\right\}^r e^{-\sum\limits_{i=0}^{r}\frac{\chi_i}{kT}}$$

or also in the known form (formula of Sakh):

$$\frac{n_{r+1}}{n_r}p_e=\frac{u_{r+1}}{u_r}2\frac{(2\pi m_e kT)^{\frac{3}{2}}kT}{h^3}e^{-\frac{\chi_{r+1}}{kT}}, \tag{15}$$

where $p_e = n_e kT$ is the partial electronic pressure.

For single ionization, equation (15) takes the form

$$\frac{z^2}{1-z}p=\frac{u_u}{u_a}2\frac{(2\pi m_e kT)^{\frac{3}{2}}}{h^3}e^{-\chi_i/kT}. \tag{16}$$

Other thermodynamic magnitudes may also be determined, for example, specific heats c_p and c_v, speed of sound, etc., but we will generally not do this and will consider these magnitudes only in the case of single ionization.

Excitation of atoms may not be considered with a great degree of accuracy ([5], page 332) and it is considered that all atoms and ions are in a normal state. As a result of this, the preceding formulas may be simplified. The sum by states for neutral atoms and ions may be replaced by constant magnitudes - statistical weights for basic conditions:

$$u_a \cong g_a, \quad u_u \cong g_u$$

and for single ionization we will have:

$$\varepsilon_T = \frac{3}{2} N_0 kT(1+z) + N_0 z\chi, \tag{17}$$

$$S = \frac{5}{2} N_0 k(1+z) + kN_0(1-z) \ln \frac{(2\pi m_a kT)^{\frac{3}{2}} g_a}{(1-z) N_0 h^3} + \\ + kN_0 z \ln \frac{(2\pi m_a kT)^{\frac{3}{2}} g_u}{zN_0 h^3} + kN_0 z \ln \frac{(2\pi m_e kT)^{\frac{3}{2}} 2}{zN_0 h^3}. \tag{18}$$

$$W = \frac{5}{2} N_0 kT(1+z) + N_0 z\chi_1. \tag{19}$$

$$F = -N_0 kT(1+z) + N_0 z\chi_1 - kN(1-z) \ln \frac{(2\pi m_a kT)^{\frac{3}{2}} g_a}{(1-z) N_0 h^3} - \\ - kN_0 \ln \frac{(2\pi m_a kT)^{\frac{3}{2}} g_a}{zN_0 h^3} - kN_0 z \ln \frac{(2\pi m_e kT)^{\frac{3}{2}} 2}{zN_0 h^3}, \tag{20}$$

$$\frac{z^2}{1-z^2} p = \frac{g_u}{g_a} 2 \frac{(2\pi m_e kT)^{\frac{3}{2}} kT}{h^3} e^{-\frac{\chi_1}{kT}}$$

or

$$\frac{z^2}{1-z} = \omega_0 \mu_H \frac{T^{\frac{3}{2}}}{p} e^{-\frac{\chi_1}{kT}}, \tag{21}$$

where constant $\omega_0 = \frac{2g_u}{g_a} \frac{(2\pi m_e k)^{\frac{3}{2}}}{h^3} m_H.$

Let us now find thermal heat capacity at constant pressure. We will proceed from the thermodynamic formula:

$$c_p = \left(\frac{\partial W}{\partial T}\right)_p.$$

We have in force (19):

$$c_p = \frac{5}{2} kN_0(1+z) + N_0 kT\left(\frac{5}{2} + \frac{\chi_1}{kT}\right)\left(\frac{\partial z}{\partial T}\right)_p. \tag{22}$$

We determine the derivative $\left(\frac{\partial z}{\partial T}\right)_p$ from formula (21), having taken from the left and right part its logarithmic derivative. We obtain:

$$\frac{2}{z(1-z^2)}\left(\frac{\partial z}{\partial T}\right)_p = \frac{1}{T}\left(\frac{5}{2}+\frac{\chi_1}{kT}\right).$$

Substituting the derivative determined by this correlation, in (22), we find the value of the specific heat at constant pressure, relative to the unit of volume:

$$c_p = \frac{5}{2}(1+z)N_0k + \frac{z(1-z^2)}{2}\left(\frac{5}{2}+\frac{\chi_1}{kT}\right)^2 N_0k \tag{23}$$

or in calculation for a unit of mass of gas:

$$C_p = \left[\frac{5}{2}+\frac{z(1-z)}{2}\left(\frac{5}{2}+\frac{\chi_1}{kT}\right)\right](1+z)\frac{R_0}{\mu_H}. \tag{24}$$

For determination of specific heat of gas with calculation of ionization in constant volume (density), we will proceed from the formula:

$$c_v = \left(\frac{\partial \varepsilon_T}{\partial T}\right)_v.$$

Formula (17) gives:

$$c_v = \frac{3}{2}N_0k(1+z) + N_0\left(\chi_1+\frac{3}{2}kT\right)\left(\frac{\partial z}{\partial T}\right)_\rho. \tag{25}$$

From formula (21), utilizing equation (2), we have:

$$\frac{2-z}{z(1-z)}\left(\frac{\partial z}{\partial T}\right)_\rho = \frac{1}{T}\left(\frac{3}{2}+\frac{\chi_1}{kT}\right),$$

which together with (25) leads to the expression:

$$c_v = \frac{3}{2}(1+z)N_0k + \frac{z(1-z)}{2-z}\left(\frac{3}{2}+\frac{\chi_1}{kT}\right)^2 kN_0 \tag{26}$$

or in calculation for a unit of mass:

$$C_v = \frac{R_0}{\mu_H}\left\{\frac{3}{2}+\frac{z(1-z)}{(2-z)(1+z)}\left(\frac{3}{2}+\frac{\chi_1}{kT}\right)^2\right\}(1+z). \tag{27}$$

The ratio of specific heats:

$$\gamma = \frac{c_p}{c_v} = \frac{\frac{5}{2} + \frac{z(1-z)}{2}\left(\frac{5}{2} + \frac{\chi_1}{kT}\right)^2}{\frac{3}{2} + \frac{z(1-z)}{(2-z)(1+z)}\left(\frac{3}{2} + \frac{\chi_1}{kT}\right)^2}. \tag{28}$$

We will now determine the adiabatic speed of sound in gas with calculation of ionization, for which we will take, as usual, the magnitude:

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s},$$

the derivative is taken with constant entropy. We will write the condition of constancy of entropy in the following form:

$$ds = 0,$$

whereto here, instead of entropy S relative to a unit of volume, we will take entropy s relative to a unit of mass, and apparently:

$$s = S/\rho.$$

Equality (18) will give:

$$\frac{3}{2}(1+z)\frac{dT}{T} + \left(\frac{3}{2} + \frac{\chi_1}{kT}\right)dz - (1+z)\frac{d\rho}{\rho} = 0.$$

Further, taking logarithmic derivations from (21) and (3), we have:

$$\frac{d\rho}{\rho} + \frac{2-z}{z(1-z)}dz = \left(\frac{3}{2} + \frac{\chi_1}{kT}\right)\frac{dT}{T};$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dz}{1+z} + \frac{dT}{T},$$

from which together with the preceding formula it follows that:

$$a^2 = \frac{p}{\rho}\,\frac{5 + z(1-z)\left(\frac{5}{2} + \frac{\chi_1}{kT}\right)^2}{3\left(1 + \frac{z(1-z)}{2}\right) + z(1-z)\left(\frac{3}{2} + \frac{\chi_1}{kT}\right)^2}. \tag{29}$$

Equating (29) and (28), we may write:

$$a^2 = \frac{2}{(2-z)(1+z)}\gamma\frac{p}{\rho}. \tag{29'}$$

All formulas of the present section are at once converted into ordinary thermodynamic correlations for monatomic ideal gas if it is assumed that z = 0 (non-ionized gas), or z = 1 (gas completely ionized), or if in general it is considered that z = const.

Further on we will consider a stream of monatomic gas in those conditions which may be limited to calculation only of single ionization. The equation of state may be taken in the form (4), and the expression for mass density of internal energy will be written thus:

$$\varepsilon_T=\frac{3}{2}\frac{R_0}{\mu_H}T(1+z)+\frac{\chi_1}{m_a}z. \tag{30}$$

In these two correlations, a function of the state of the gas entered - the degree of ionization z, which we will determine with the aid of equation (21).

Thus, in all our reasonings, we consider gas ideal in a thermodynamic relation, but we take into account its own kind of chemical reaction - ionization, which takes place in a considered gas mixture.

Section 5. One-Dimensional Movement with Influx of Heat Due to Radiation

In this section is considered the simplest problem of hydromechanics with calculation of radiation, namely: stationary rectilinear movement of ideal gas. For simplification of the problem, thermal conductivity and viscosity are disregarded.

Gas is considered monatomic so as to make it possible to not consider dissociation of molecules at high temperatures, and also so as to not calculate the coefficient of absorption in regions of a molecular spectrum. In addition, we utilize the hypothesis concerning local thermodynamic equilibrium, i.e., we consider that although the temperature at various points of the stream varies, at each point there is equilibrium of all the considered processes in the temperature peculiar to this point. This spares us the necessity of considering the time of relaxation (for example, of the process of ionization). Single ionization is considered.

In parallel are adduced results for homogeneous gas, i.e., for ideal gas [6], the state of which throughout all regions of the flow of gas does not change (therein, for example, the degree of ionization). Correlations for description of radiation are taken in mean form by frequencies and by directions, whereby dispersion will be disregarded. Temperatures are considered to be not so high as to consider mechanical action of a radiation field (light pressure), and speeds not so great as to take into account relativity effects.

We will consider justified for our case Kirchhoff's law (19), section 1).

The system of hydrodynamic equations (8) - (10) section 3 take the form:

$$\left.\begin{aligned} \rho u &= m, \\ p + mu &= n, \\ m\varepsilon_T + nu - \frac{mu^2}{2} + H &= l. \end{aligned}\right\} \tag{1}$$

Here it is also necessary to add equations (17), (30), and (16) from section 4, and also the equation of transfer of radiation (13, section 3). Assuming that there is approximately fulfilled the equality:

$$\int_{4\pi} I \cos^2\theta \, d\Omega = \frac{1}{3} \int_{4\pi} I \, d\Omega,$$

equation (13, section 3) together with (15, section 3) may be reduced to an ordinary differential equation:

$$\frac{d^2H}{d\tau^2} = \frac{H}{\alpha_0^2} + \frac{8\sigma T^3}{\alpha_0'} \frac{dT}{d\tau}, \tag{2}$$

σ is the constant of Stephan - Boltzmann $\alpha_0 = \frac{1}{\sqrt{3}}$, $\alpha_0' = \frac{1}{2}$. There exist other methods of averaging which also lead to equation (2), but with other values of α_0 and α_0'. Thus, according to Mustel' [7], $\alpha_0 = \frac{2}{3}$, $\alpha_0' = \frac{2}{3}$; and according to Schwarzschild, $\alpha_0 = \frac{1}{2}$, $\alpha_0' = \frac{1}{2}$.

We will introduce the unmeasured variables:

$$\left.\begin{aligned} &\tilde{p}=\frac{p}{p_1}, \quad \tilde{\rho}=\frac{\rho}{\rho_1}, \quad \tilde{u}=\frac{u}{u_1}, \quad \tilde{T}=\frac{T}{T_1}; \\ &\tilde{H}=\frac{H}{\rho_1 u_1^3}, \quad \tilde{\varepsilon}_T=\frac{\varepsilon_T}{u_1^2}, \end{aligned}\right\} \tag{3}$$

where by the index 1 are designated values of variables in any point $\tau = \tau_1$. All functions entering into systems (1) and (2) may be expressed by one of these in an unmeasured form, for example, by unmeasured speed $\tilde{u}$, which is equal in force of an equation of indissolubility to the value of the unmeasured specific volume $\tilde{v} = \frac{v}{v_1}$:

$$\left.\begin{aligned} &\rho=\tilde{u}^{-1}, \\ &\tilde{p}=q_1(\beta_1-\tilde{u}), \\ &\tilde{T}=\frac{1+z_1}{1+z} q_1(\beta_1-\tilde{u})\tilde{u}, \\ &\tilde{H}=2(\tilde{u}-1)(\tilde{u}-\beta_8)+\beta_5(z_1-z)+\tilde{H}_1, \\ &\tilde{\varepsilon}_T=\frac{3}{2}(\beta_1-\tilde{u})\tilde{u}+\beta_5 z. \end{aligned}\right\} \tag{4}$$

For a homogeneous thermally and calorifically perfect gas ($\varepsilon_T = c_v T$, $p = \frac{R}{\mu}\rho T$, $z = z_1' = \text{const}$, $\alpha^2 = \gamma\frac{p}{\rho}$; $\gamma = \frac{c_p}{c_v}$, wherein c_v, γ, R are constant magnitudes) the last three equations of the system (21) take the form:

$$\left.\begin{aligned} &\tilde{T}=q_1(\beta_1-\tilde{u})\tilde{u}, \\ &\tilde{H}=\frac{1}{2}\frac{\gamma+1}{\gamma-1}(\tilde{u}-1)(\tilde{u}-\beta_8)+\tilde{H}_1, \\ &\tilde{\varepsilon}_T=\frac{1}{\gamma-1}(\beta_1-\tilde{u})\tilde{u}. \end{aligned}\right\} \tag{4'}$$

In equations (4) and (4'):

$$\left.\begin{aligned} &q_1=\frac{\rho_1 u_1^2}{p_1}, \\ &\beta_1=1+\frac{1}{q_1}, \\ &\beta_8=\frac{\gamma-1}{\gamma+1}+\frac{2\gamma}{\gamma+1}\frac{1}{q_1}, \\ &\beta_5=\frac{\chi_1}{m_a u_1^2}, \end{aligned}\right\} \tag{5}$$

whereto for monatomic gas $\gamma = \frac{5}{3}$. For homogeneous gas $q_1 = \gamma M_1^2$, where M_1 is the number M for conditions of flow at point $\tau = \tau_1$.

For the degree of ionization z, from (16, section 4) we have:

$$\frac{z^2(1+z)}{1-z}=\omega_0'(\beta_1-\widetilde{u})^{\frac{3}{2}}\widetilde{u}^{\frac{5}{2}}e^{-\frac{\beta_3(1+z)}{\widetilde{u}(\beta_1-\widetilde{u})}} \tag{6}$$

here

$$\omega_0'=(1+z_1)^{\frac{5}{2}}q_1^{\frac{3}{2}}\frac{2g_u}{g_a}\frac{(2\pi m_e kT_1)^{\frac{3}{2}}kT_1}{h^3p_1}.$$

Placing (4) in (2), we obtain the fundamental equation of the problem:

$$a_1\frac{d^2\widetilde{u}}{d\tau^2}+a_2\left(\frac{d\widetilde{u}}{d\tau}\right)^2+a_3\frac{du}{d\tau}+a_4=0, \tag{7}$$

where

$$a_1=\widetilde{u}-\beta_4-\frac{\beta_5}{4}\frac{dz}{d\widetilde{u}},\quad \beta_4=\frac{5}{8}\beta_1,$$

$$a_2=1-\frac{\beta_5}{4}\frac{d_2z}{d\widetilde{u}^2},$$

$$a_3=2\beta_2\frac{(\beta_1-\widetilde{u})^3\widetilde{u}^3}{(1+z)^4}\left(u-\frac{\beta_1}{2}+\frac{1}{2}\frac{u(\beta_1-\widetilde{u})}{1+z}\frac{dz}{du}\right),$$

$$a_4=-\frac{1}{2\alpha_0^2}\left[(\widetilde{u}-1)(\widetilde{u}-\beta_3)+\frac{\beta_5}{2}(z_1-z)+\frac{1}{2}\widetilde{H}_1\right],$$

$$\beta_4=\frac{2\sigma T_1^4}{\alpha_0^1\beta_1u_1^3}q_1^4(1+z_1)^4$$

(for homogeneous gas, in the first part of the expression for β_2, instead of the coefficient 2 is entered $8\,\frac{\gamma+1}{\gamma-1}$).

If we introduce a new unknown function

$$y=\frac{d\widetilde{u}}{d\tau}, \tag{8}$$

then we obtain an ordinary differential equation of the first order:

$$a_1y\frac{dy}{d\widetilde{u}}+a_2y^2+a_3y+a_4=0. \tag{9}$$

This differential equation has the integral:

$$\widetilde{u}=\widetilde{a}^*=\beta_4+\frac{1}{4}\beta_5\frac{dz}{d\widetilde{u}} \tag{10}$$

and six special points. Two of these lie on the axis y = 0:

Point A, $$\widetilde{u}=\frac{1-\beta_3}{2}+\sqrt{\frac{1}{4}(1-\beta_3)^2+\frac{\beta_5}{2}(z_A-z_1)-\frac{H_1}{2}}, \tag{11}$$

Point B, $$\widetilde{u}=\frac{1+\beta_3}{2}-\sqrt{\frac{1}{4}(1-\beta_3)^2+\frac{\beta_5}{2}(z_B-z_1)-\frac{H_1}{2}}, \tag{12}$$

where z_A and z_B are values of z with corresponding values of $\widetilde{u}$. These points correspond to stationary values of the function $\tilde{u}(\tau)$, i.e., to conversion of acceleration to zero, and are saddles according to the classification of Poincare [8]. Integral curves in the vicinity of these points have the form:

$$\eta - w_2\xi = C|\eta - w_1\xi|^{\frac{w_1}{w_2}}, \tag{13}$$

where

$$w_1 = -\frac{1}{2}\frac{a_3^0}{a_1^0} \pm \sqrt{\frac{1}{4}\left(\frac{a_3^0}{a_1^0}\right)^2 + \frac{1}{a_0^2}}$$

(zeros on corresponding symbols designate that their values are taken at special point A or correspondingly B), and two integral curves

$$\eta = w_1\xi, \quad \eta = w_2\xi \tag{14}$$

pass through special point A (correspondingly B).

[Translator's note: Page 101 of original not available].

.... Investigation of the structure of a shock wave in this case should be done with constant values of all physical characteristics entering into fundamental equations describing the process (coefficient of viscosity and thermal conductivity, specific heats, etc.). Then there may be a case in which a change of conditions in the gas produces physical and chemical reactions in it (for example, excitation of rotary degrees of freedom in the gas and thereby change of specific heat). But these reactions may occur quite slowly as compared with the speed of compression of gas in a shock wave. Then, the entire process may be divided into two stages: at first occurs the compression of gas without calculation of a reaction which interests us, and then a reaction occurs in the gas compressed by a wave. If the speed of the reaction is sufficiently great so that inside of the compression wave it succeeds in being substantially manifested, then investigation of the structure of the compression wave should proceed from calculation of this reaction. It becomes necessary to consider relaxation processes and to enlist laws of chemical kinetics. Calculations become very labor-consuming, and from ordinary gas dynamics it becomes necessary to go far into the field of statistical mechanics and kinetics.

Investigation is significantly simplified if the hypothesis concerning local equilibrium may be utilized; i.e., if it is assumed that at each point at each moment of time there is equilibrium in temperature and pressure peculiar to that point and to that moment of time for all processes important in the problem. Such calculations represent an investigation of a maximum instance, and sometimes may give an approximate picture of a phenomenon or in any case a qualitative representation of the structure of the wave.

In an ideal (in a hydrodynamic sense) fluid, the compression wave is a geometrical surface devoid of thickness, and a corresponding hydrodynamic problem concerning the internal structure of a wave leads to a system of algebraic equations, solution of which is the correlation of Rankin-Hugoniot. In a compression wave, temperature, pressure, density, etc., change intermittently. A model of a perfect non-viscous gas with constant thermal heat capacities cannot even qualitatively reflect phenomena taking place inside of a compression wave. If then are introduced into these equations members containing derivatives, then it will be possible to obtain a continuous change of mechanical magnitudes characterizing the stream. This may be done by way of calculation of viscosity and thermal conductivity [9]: in the first integrals of the fundamental system of equations enters the derivative $\frac{du}{dx}$, due to work of forces of viscosity, and the magnitude $\frac{dT}{dx}$, characterizing transfer of heat by way of thermal conductivity. Continuous solution is obtained in that case if transfer of heat with the aid of radiation is considered, since corresponding dependence introduce into the system of first integrals differential correlations even if viscosity and thermal conductivity are disregarded.

It may be said beforehand that if a weak compression wave is diffused in gas under ordinary conditions, then effect of radiation on the structure of the wave is so small as to be disregarded. If however the compression wave is very intensive, which is accompanied by strong heating of the gas, then radiation begins to play an important, and it may be under some conditions, dominating role as compared with viscosity and thermal conductivity, since, generally speaking, transfer of heat by thermal conductivity is proportional to the gradient of temperature, and radiation transfer of heat - to the gradient of the fourth degree of temperature. Further on we will consider such a case as is applicable to non-viscous and non-heat-conducting gas as set forth in the preceding section. In practical computations are utilized the formula of Sakh for determination of degree of ionization and the formula of Chandrasekar for computation of the coefficient of opacity.

We will limit ourselves to study of the structure of a plane compression wave. The problem leads to investigation of equation (8) of the preceding section with the following boundary condition:

$$x = \pm\infty, \quad y = 0 \quad (\tau = \pm\infty), \tag{1}$$

i.e., we will consider that the stream extends on both sides to infinity, and that at great distances from the beginning, derivatives of all variables (p, ρ, T, u, etc.) in accordance with x (correspondingly in accordance with τ) become zero. Moreover, insofar as in infinitely distant regions T = const (thermodynamic equilibrium), then here the radiation should be isotropic, as a result of which

$$H_{x=\pm\infty} = 0. \tag{2}$$

Let us assume that movement occurs on the side of the positives x. We assume $\tau_1 = -\infty$. We designate the values of thermodynamic values in point $\tau = +\infty$ with the index 2. From (1, section 5) we obtain the correlation of Rankin-Hugoniot for a compression wave, and solving those, for example, relative to $\tilde{\rho}_2$, we find the equation of shock adiabat[1]:

$$\rho_2 = \frac{1 + 4\tilde{p}_2}{4 + \tilde{p}_2 - 2\frac{\chi_1}{kT_1}\frac{z_2 - z_1}{1 + z_1}}, \tag{3}$$

which converts to the ordinary equation of adiabat of Rankin - Hugoniot for monatomic gas if the gas is homogeneous:

$$\tilde{\rho}_2 = \frac{1 + 4\tilde{p}_2}{4 + \tilde{p}_2}. \tag{3'}$$

---

1. For a more general case in which is considered mechanical action of radiation and there are no means by frequencies and directions, the demand of presence of thermodynamic equilibrium in infinity leads to the terms:

Equation (3) differs from equation (3') by the presence in the denominator of a member dependent not only on $\tilde{p}_2$ but also on the initial state of the gas (initial temperature and pressure) and on physical properties of the gas (atomic mass and potential of ionization). For a compression wave, always $z_2 \geq z_1$, and moreover $(\chi_1/kT_1) > 0$, and therefore adiabat (3) lies higher than adiabat (3'), having with it two common points in $\tilde{p}_2 \to 1$ and in $\tilde{p}_2 \to \infty$. Between these points lies one extreme (maximum) of the function $\tilde{\rho}_2(\tilde{p}_2)$. If we take into account repeated ionization of complex atoms, then the number of such maximums will be equal to the number of considered degrees of ionization.

We may write the dependence of all other magnitudes on $\tilde{p}_2$ (Figures 3 to 5), for example:

$$\tilde{T}_2 = \frac{1+z_1}{1+z_2} \frac{4+\tilde{p}_2 - 2\frac{\chi_1}{kT_1}\frac{z_2-z_1}{1+z_1}}{1+4\tilde{p}_2}; \tag{4}$$

$$q_1 = \frac{(\tilde{p}_2-1)(1+4\tilde{p}_2)}{3(\tilde{p}_2-1) - 2\frac{\chi_1}{kT_1}\frac{z_2-z_1}{1+z_1}}, \tag{5}$$

or on any other parameter, for example, on the number M_1, i.e., on the number M in a medium undisturbed by a wave, as related to the speed of diffusion of a wave (Figures 6 to 9).

$$x = -\infty, \quad H_{x_1} = H_x = 0, \quad p_{Rxx} = -\frac{\epsilon_R}{3} = \frac{aT_1^4}{3};$$

$$x = +\infty, \quad H_{x_1} = H_x = 0, \quad p_{Rxx} = -\frac{\epsilon_R}{3} = \frac{aT_2^4}{3}.$$

Proceeding from equations (8 - 10, section 3), we now obtain the generalized terms of Rankin - Hugoniot for a compression wave in the following form:

$$\tilde{\rho}_2\tilde{u}_2 = 1,$$

$$q_1(1-\tilde{u}_2) - (\tilde{p}_2+\tilde{p}_{R2}) + 1 + \tilde{p}_{R1} = 0,$$

$$\frac{q_1}{2}(1-\tilde{u}_2^2) + \tilde{\epsilon}_{T_1} - \tilde{\epsilon}_{T_2} + 4(\tilde{p}_{R_1} - \tilde{u}_2\tilde{p}_{R_2}) - \tilde{p}_2\tilde{u}_2 + 1 = 0.$$

Here is designated $\tilde{p}_R = \frac{p_R}{p_1}$. These terms are easily obtained, also not resorting to common equations (Tgomas L. H., Journal Chem. Phys., 1944, v. 12, p. 449).

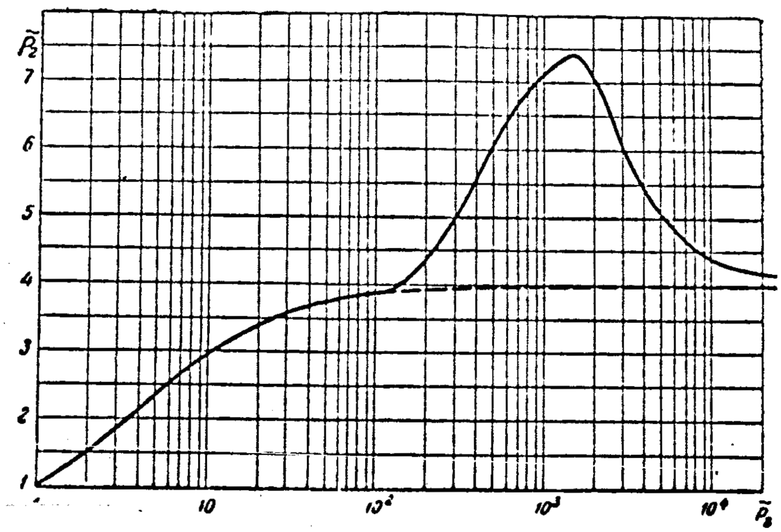

Figure 2. Shock adiabat in monatomic hydrogen ($p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute) with calculation of ionization (solid line) and without calculation of ionization (dotted line).

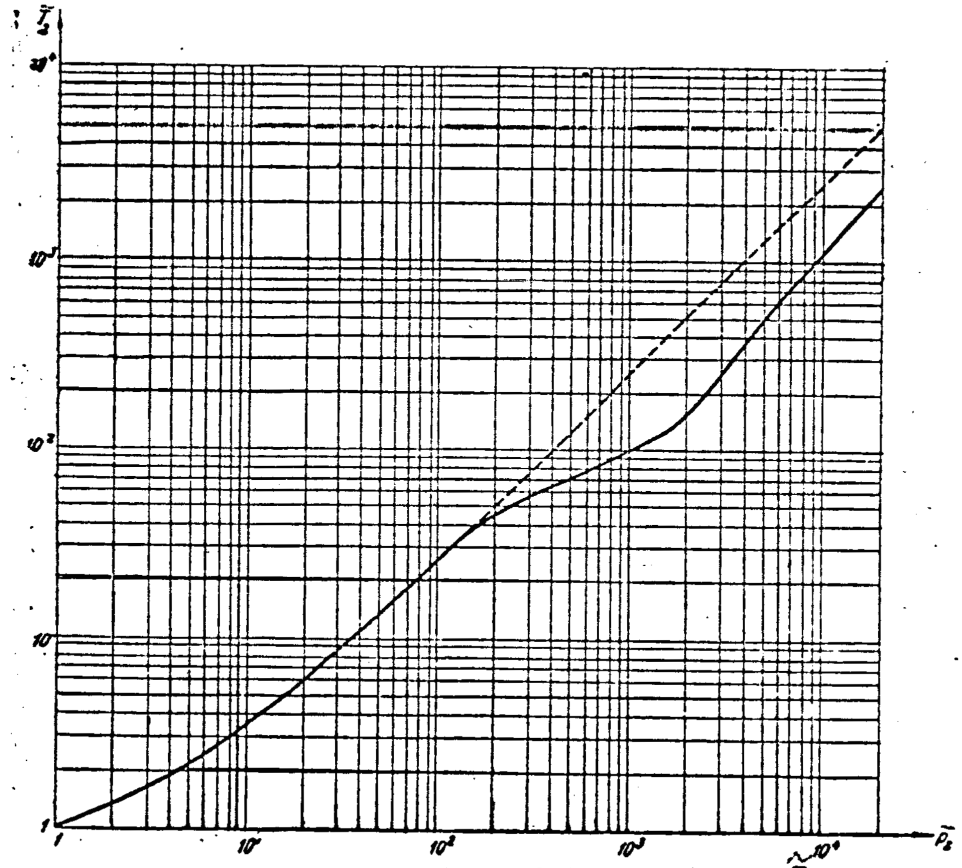

Figure 3. Dependence of the relation of temperatures $\tilde{T}_2 = T_2/T_1$ of a compression wave in monatomic hydrogen on the relation of pressures $\tilde{p}_2 = p_2/p_1$ (conditions the same for Figure 2). Dotted lines are related to homogeneous gas ($z_1 = z_2 = 0$).

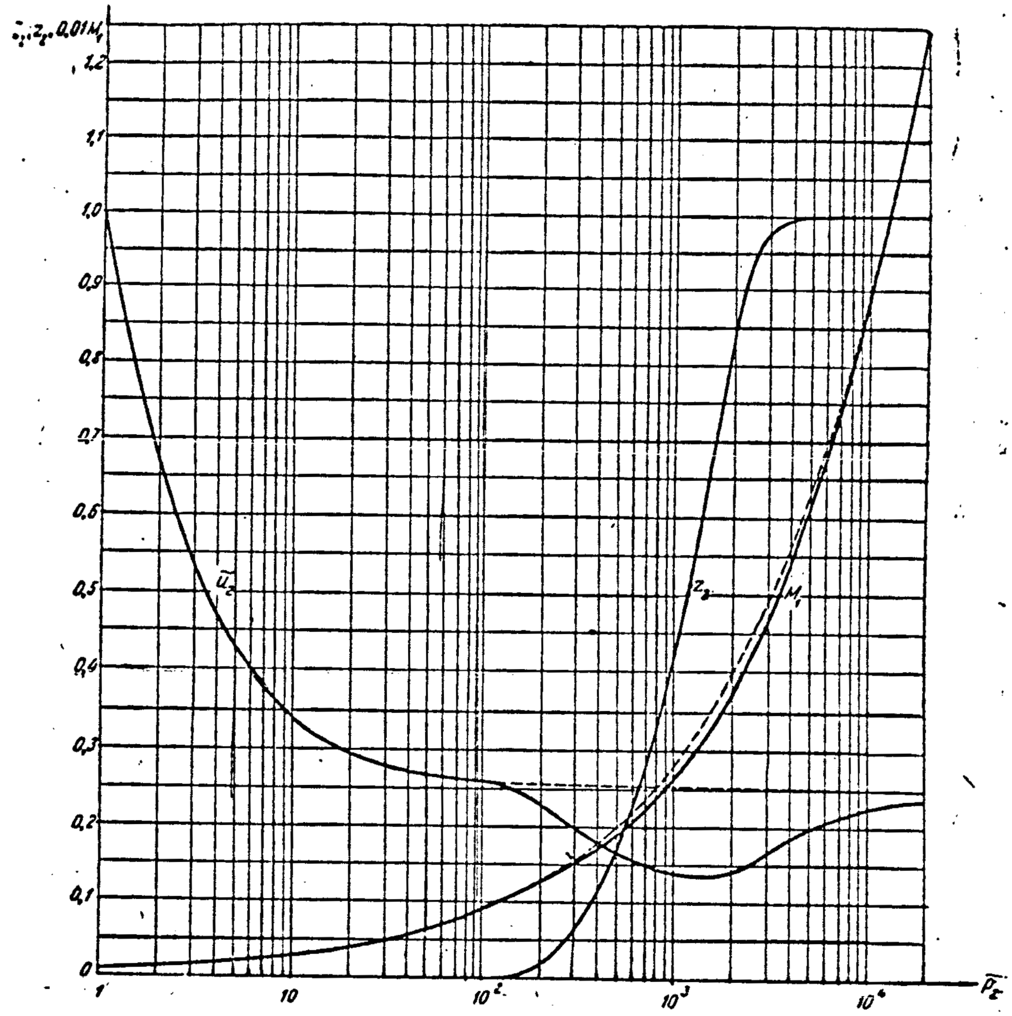

Figure 4. Relation of speeds $\tilde{u}_2$ of a compression wave, the degree of ionization z_2, and the number M_1 (number M before the wave) in dependence on the relationship of pressures $\tilde{p}_2$ (conditions the same as for Figure 2). Dotted lines are related to homogeneous gas.

Let us consider the change of entropy inside a compression wave. Proceeding from the known thermodynamic formula:

$$\frac{ds}{dt} = \frac{q}{T}$$

(s is entropy, t is time), we find:

$$d\tilde{s} = -\frac{4(1+z)}{(\beta_1-\tilde{u})\tilde{u}}\left(\tilde{u}-\beta_4-\frac{\beta_5}{4}\frac{dz}{d\tilde{u}}\right)d\tilde{u}; \quad \tilde{s}=\frac{s}{R}. \tag{6}$$

Entropy at first increases, reaches a maximum in point $\tilde{u} = \tilde{a}^*$, then decreases. A complete change of entropy in passage through a wave equals:

$$\Delta\tilde{s} = \tilde{s}(\tilde{u}_2) - \tilde{s}(1) = -4q_1(1+z_1)\int_1^{\tilde{u}_2} \frac{a_1(\tilde{u})}{\tilde{T}(\tilde{u})}\,\tilde{d}u. \tag{7}$$

This magnitude is positive, i.e., the second law of thermodynamics is fulfilled if it is applied to beginning and final states of the system.

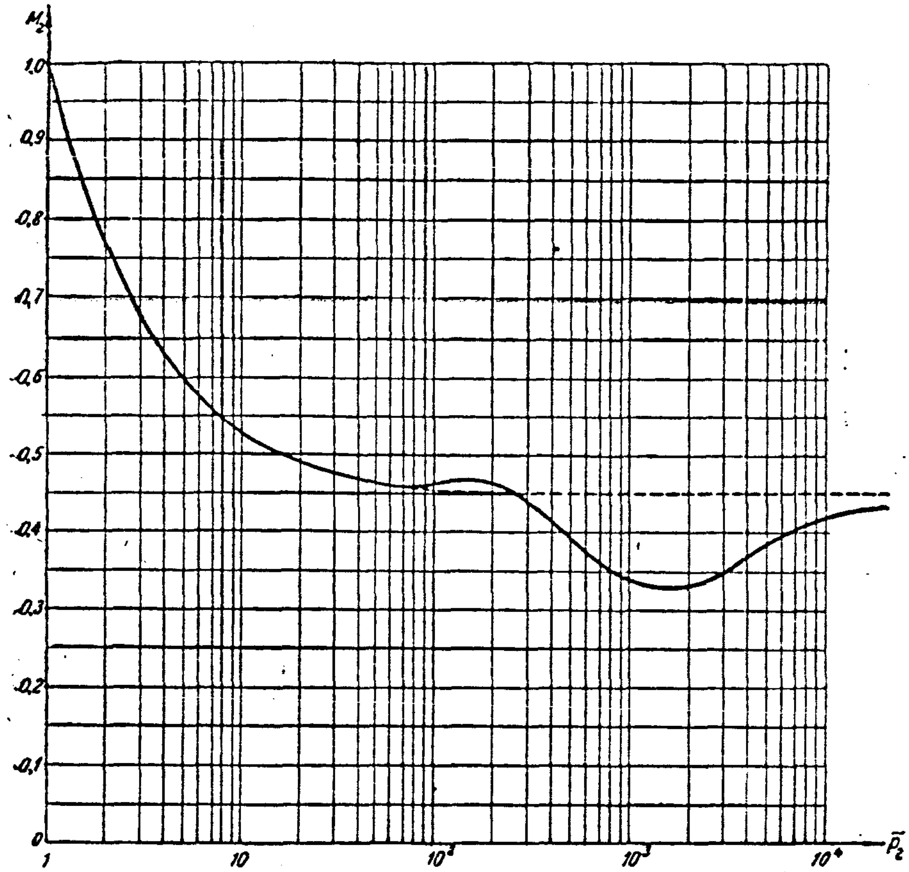

Figure 5. Dependence of the number M_2 (number M after a wave) on the relation of pressures $\tilde{p}_2$ (conditions the same as for Figure 2). Dotted lines are related to homogeneous gas.

It is not difficult to see that in the case of flow through a compression wave, special points A and B correspond to infinity to the left and right, and in point A we have $\tilde{u} = \tilde{u}_1 = 1$, in point B $\tilde{u} = \tilde{u}_2$, whereto $\tilde{u}_2 \leq \tilde{a}^* \leq 1$ (sign of equality only in a case of movement with constant speed equal to the speed of sound), and the speed diminishes from 1 (supersonic zone) to $\tilde{u}_2$ (subsonic zone). We will consider $\tilde{u}$ a monotonously decreasing function of τ (correspondingly of x). Consequently, the problem of study of flow inside a compression wave resolves itself to search of the solution of the equation (8, section 5) satisfying the conditions:

$$\left.\begin{aligned}\tilde{u}=1, \quad y=0;\\ \tilde{u}=\tilde{u}_2, \quad y=0\end{aligned}\right\} \tag{8}$$

and disposed in the lower semiband (Figure 10). The solution should pass through points A and B and intersect the integral curve (10, section 5), which is possible only in special points C, D, or F, and will consist of two branches. The left branch (subsonic) in any case comes from point B and asymptotically the line $\tilde{u} = \tilde{a}*$. The right branch

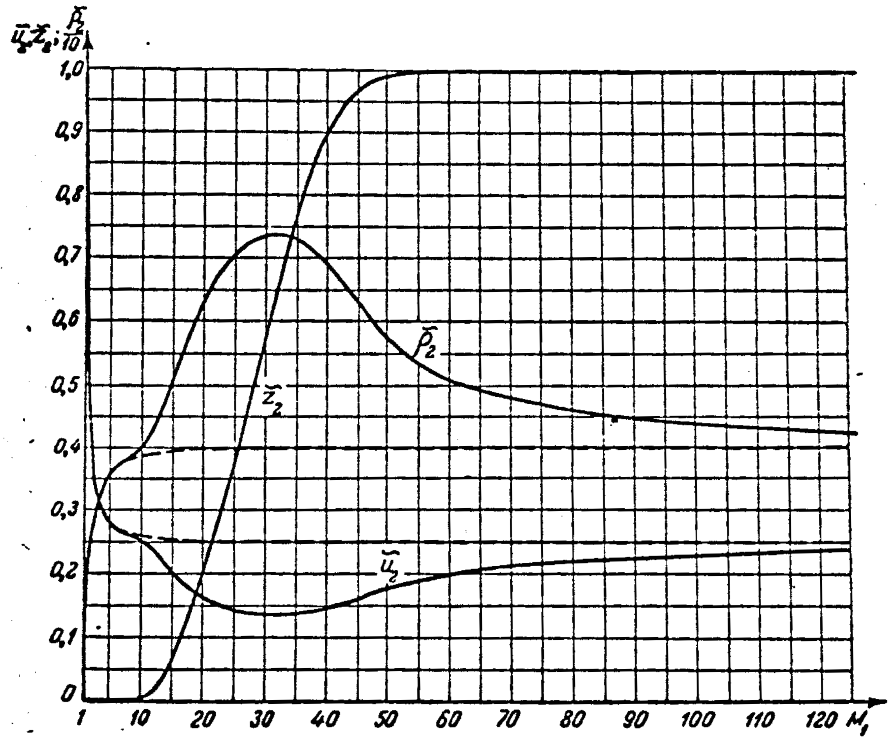

Figure 6. Relation of speeds $\tilde{u}_2$ and densities $\tilde{p}_2$ of a compression wave and the degree of ionization behind a wave z_2 in dependence on the number M_1 (conditions the same as for Figure 2). Dotted lines are related to homogeneous gas.

(supersonic) comes from point A and approximates the line $\tilde{u} = \hat{a}*$ if $K* > 1$ and intersects it at point C if $K* < 1$; in the first case in passage through the speed of sound, the line $\tilde{u}\,(\tau)$ has a vertical tangent, and in the second case, an angular point.

In both cases there is a continuous monotonously decreasing function $\tilde{u}(\tau)$, and consequently also $\tilde{u}(x)$, satisfying all provided conditions, whereas without calculation of radiation, system (1) gives two constant values $\tilde{u}$ ($\tilde{u} = 1$, $\tilde{u} = \tilde{u}_2$), i.e., the solution is discontinuous. With calculation of radiation is obtained a continuous solution, but with some peculiarity, mentioned above, in passage through the speed of sound and connected with those assumptions made by us (in particular, with disregard of viscosity and thermal conductivity).

The member $a_3 y$ of equation (7, section 5) contains a small multiplier σ, and in some cases this may be disregarded (for example, with diffusion of a wave in atomic hydrogen under ordinary conditions this may be done for numbers M_1 of the order 15 and even greater). Then the equation may be integrated, and for the adopted boundary conditions we obtain:

$$\tau = \pm \alpha_0 \ln \left(\frac{a_4}{a_4^*} \right) \tag{9}$$

(plus sign for $\tilde{u} < \tilde{a}^*$, and minus sign for $\tilde{u} > \tilde{a}^*$). Equation (7, section 5) with $\beta_2 = 0$ has four special points A, B, E, and F. The equation does not have special points C and D in this case, although the line $\tilde{u} = \tilde{a}^*$ remains an integral curve. Therefore, the approximation mentioned above is possible only with $K < 1$.

Dependence of hydrodynamic parameters on $\tilde{u}$ is given by equalities obtained in section 5, if it is assumed that $H_1 = 0$, and it is shown for several examples in Figure 11 to 16. For one and the same value q_1 between magnitudes obtained without calculation of ionization (we will designate them with the index N) and magnitudes with calculation of ionization, are justified for one and the same value $\tilde{u}$ the correlations:

$$\tilde{\rho} = \tilde{\rho}_N, \quad \tilde{p} = \tilde{p}_N,$$
$$\tilde{T} = \frac{1 + z_1}{1 + z} \tilde{T}_N,$$
$$\tilde{H} = \tilde{H}_N - \beta_6 (z - z_1).$$

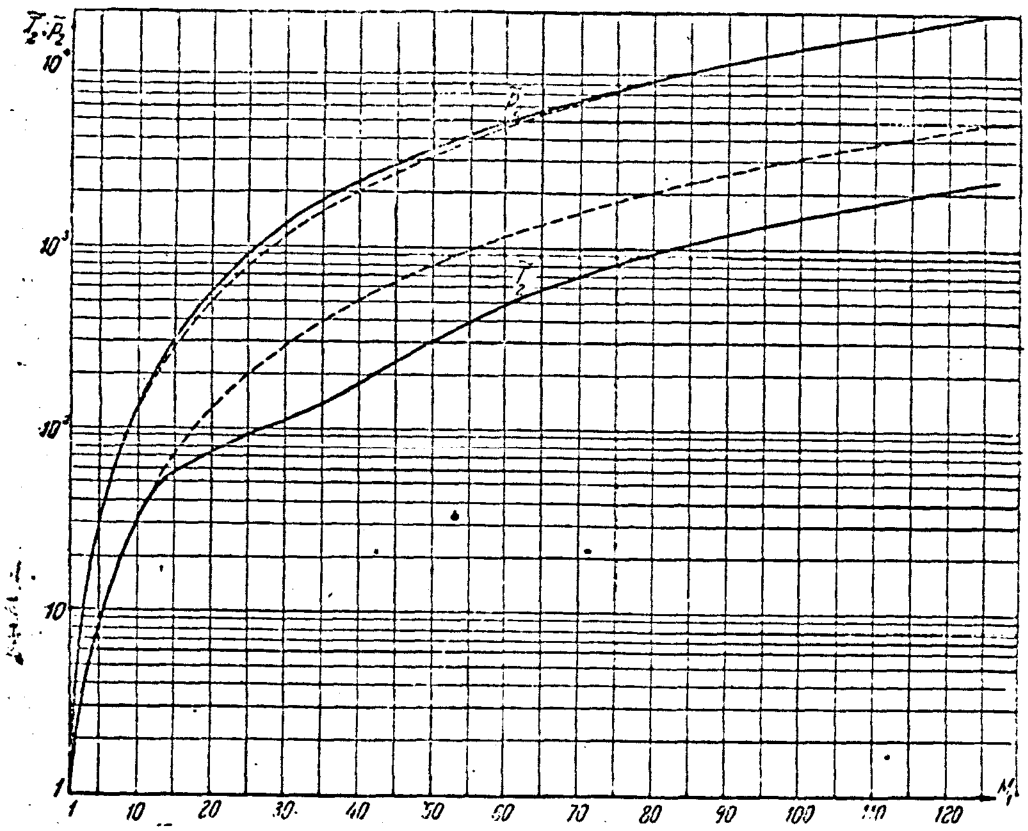

Figure 7. Relations of pressures and temperatures of a compression wave in dependence on the number M_1 (conditions the same as for Figure 2). Dotted lines are related to homogeneous gas.

Ionization leads to increase of speed $\tilde{u}_2$ after a wave and to decrease of temperature and absolute value of the stream of radiation. It is not difficult to see that the maximum value of relative temperature is obtained at the point

$$\tilde{u} = \tilde{u}_m = \frac{2\beta_1}{4 + z_m(1 - z_m)}, \tag{10}$$

where z_m is determined by the formula of Sakh for this value $\tilde{u}$. Apparently, $\frac{8}{17} \leqslant \frac{u_m}{\beta_1} \leqslant \frac{1}{2}$, whereto the greatest value of $\tilde{u}_m$ corresponds to homogeneous gas, and the least - to such initial conditions in the gas as when $z_m = ½$. The maximum value of $\tilde{T}_m$ is included in the limits $\frac{48}{289} \leqslant (\tilde{T}_m/q_1\beta_1^2) \leqslant \frac{1}{4}$.

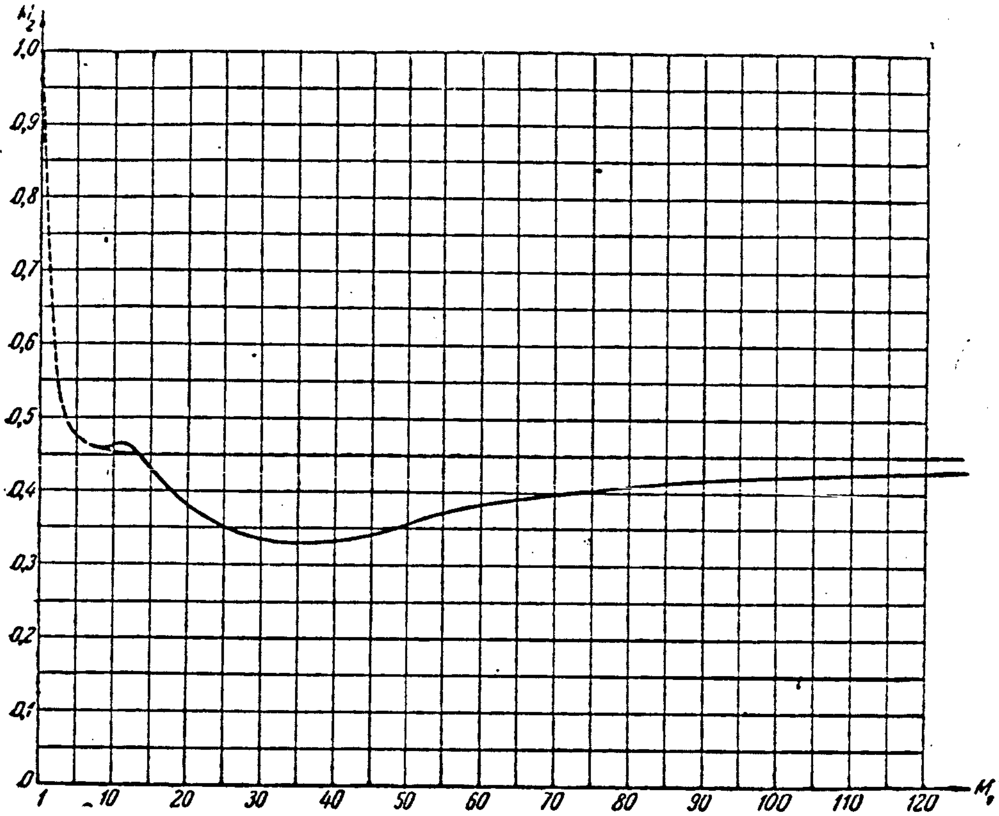

Figure 8. Dependence of the number M_2 after a compression wave with calculation of ionization (M_{2z}) and for homogeneous gas (dotted line) on the number M_1 (conditions the same as for Figure 2).

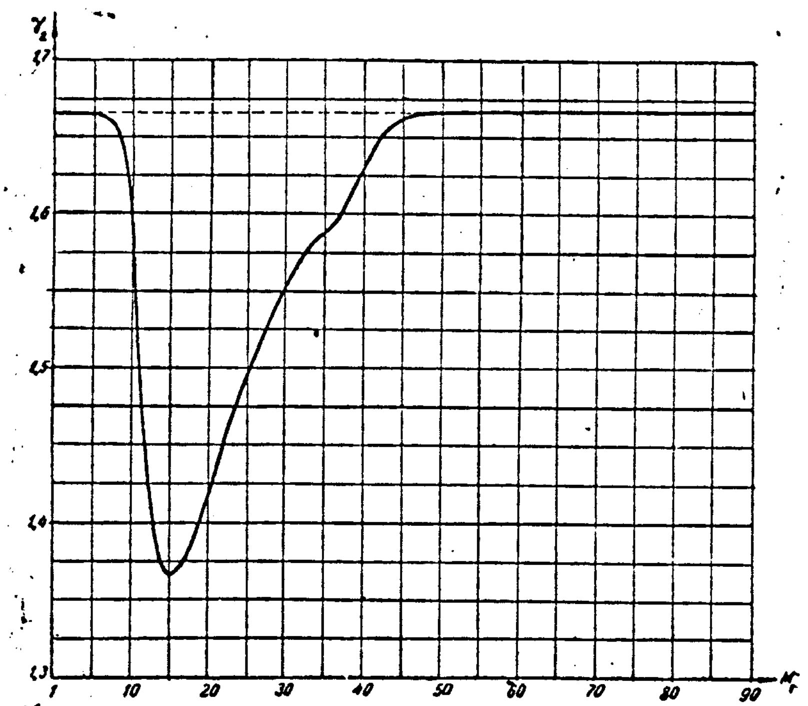

Figure 9. Dependence of the relation of specific heats γ_{z2} on the number M_1 (conditions the same as for Figure 2). Dotted straight line related to homogeneous gas.

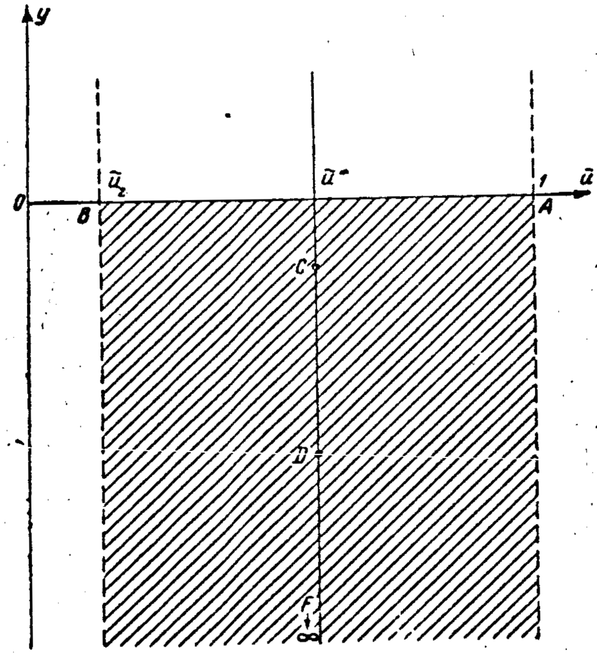

Figure 10. Region of change y in a problem concerning structure of a compression wave.

The critical speed $\tilde{u}^* = \tilde{a}^*$ for homogeneous gas equals $\frac{5}{8}\beta_1$, with calculation of ionization, as is not difficult to calculate:

$$\tilde{a}^* = \frac{\beta_1}{1 + \frac{2 + z(1-z)}{2\gamma}}, \tag{11}$$

and from here it follows, if it is considered that $1 \leq \gamma \leq \frac{5}{8}$

$$\frac{8}{17} \leqslant (\tilde{a}^*/\beta_1) \leqslant \frac{5}{8},$$

i.e., in the presence of ionization critical speed is less than the critical speed of homogeneous gas in one and the same conditions to infinity on the left. Equating (10) with (11), we find that $\tilde{u}_m \geq \tilde{a}^*$, whereto the sign of equality is justified with $\gamma = 1$, or with $\gamma = \frac{5}{3}$ (movement of gas everywhere with the speed of sound); in all other cases we have inequality. Moreover, it is easy to calculate that with $q_1 > \frac{5}{3}$ we have $\tilde{u}_m > \tilde{u}_2$. Consequently, maximum temperature is attained not "after the wave," i.e., not in infinity to the right, but "inside the wave," in the subsonic zone.

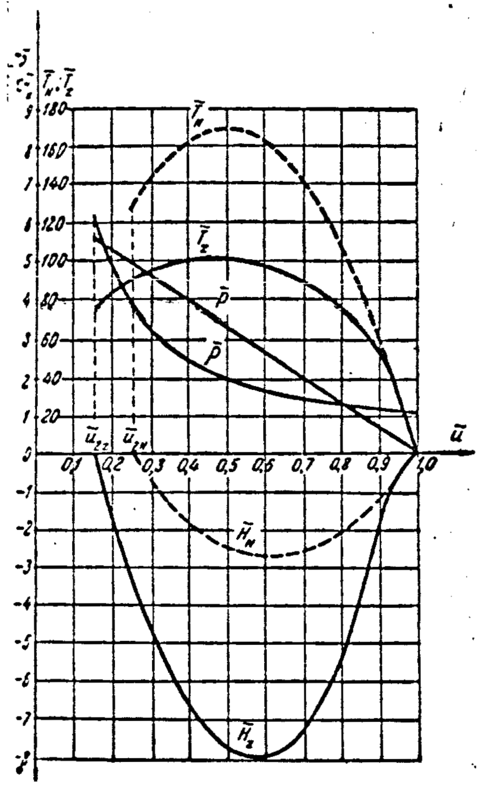

Figure 11. Relative pressure $\tilde{p}$, temperature $\tilde{T}$, density $\tilde{\rho}$, and stream of radiations H along axis x in dependence on unmeasured speed $\tilde{u}$ inside of a compression wave with $M_1 = 20$. With indexes z and N are designated corresponding values with calculation of ionization and without calculation of of ionization (homogeneous gas).

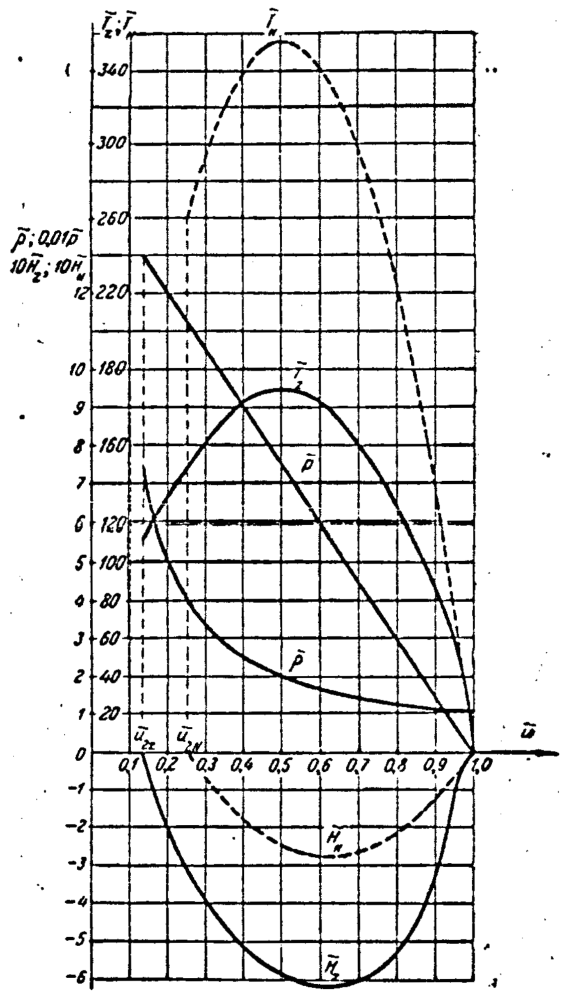

Figure 12. Relative pressure $\tilde{p}$, temperature $\tilde{T}$, density $\tilde{\rho}$, and stream of radiation H along axis x in dependence on unmeasured speed $\tilde{u}$ inside of a compression wave with $M_1 = 30$. With indexes z and N are designated corresponding values with calculation of ionization and without calculation of ionization (homogeneous gas).

The degree of ionization z may change from 0 to 1 and, as is seen from formula (6, section 5) and in Figures 17 and 18, reaches the maximum in the region of subsonic speeds. If conditions are near normal, then with numbers $M_1 < 6$ for such gas as monatomic hydrogen, ionization may be disregarded in equation (7, section 5). Actually, in this case, maximum magnitudes of the degree of equilibrium ionization and of

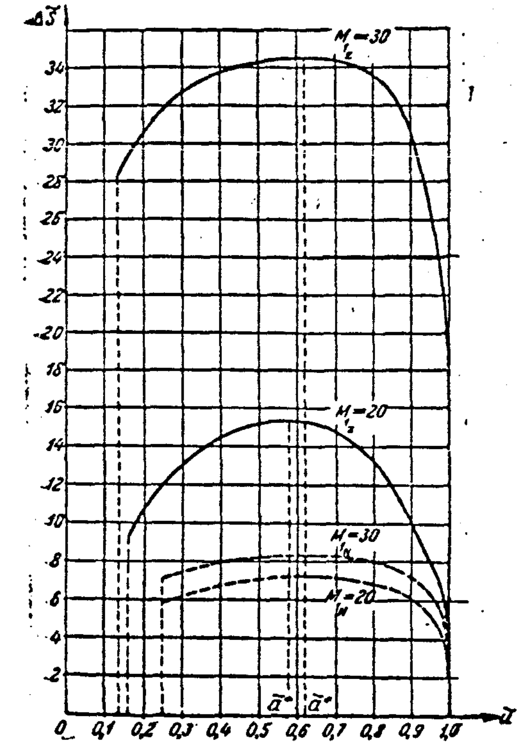

Figure 13. Change of unmeasured value of entropy $\Delta\tilde{s}=\tilde{s}(u)-\tilde{s}(1)$ with change of unmeasured speed $\tilde{u}$ inside of a compression wave with calculation of ionization (solid curves) and without calculation of ionization (dotted curves) for $M_1 = 20$ and $M_1 = 30$.

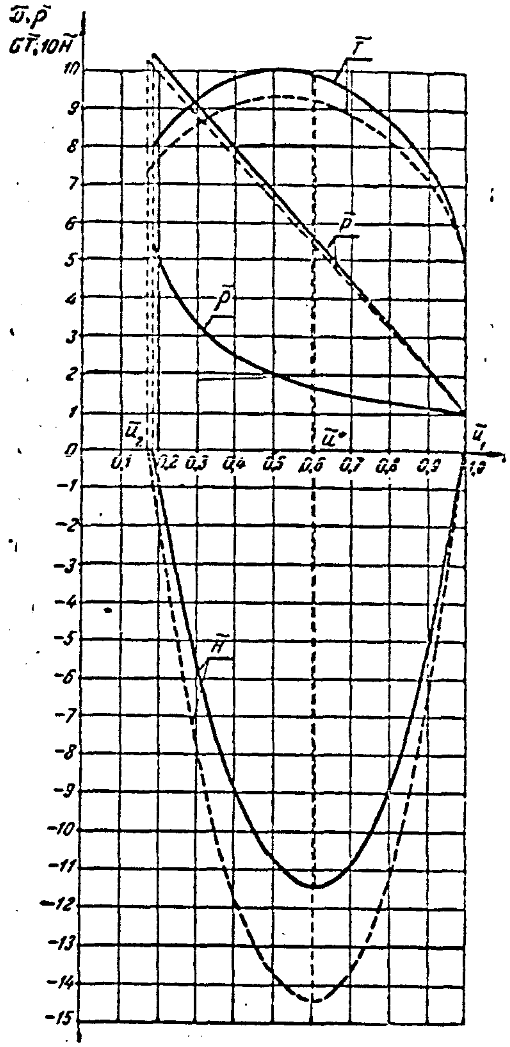

Figure 14. Relative pressure $\widehat{p}$, temperature $\tilde{T}$, density $\tilde{\rho}$, and stream of radiation $\widehat{H}$ along axis x in dependence on unmeasured speed u inside compression wave in monatomic hydrogen (solid curves) and in argon (dotted curves); $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

moduli derived inside the wave prove to be very small as compared with unity, as if seen from Figure 19 and from the table on page 49.

<u>Measured coordinate</u>. Until now we have utilized (16, section 3), which afforded the possibility to completely not touch upon characteristics of the function $\alpha(\tilde{u})$. In order to go over to a measured

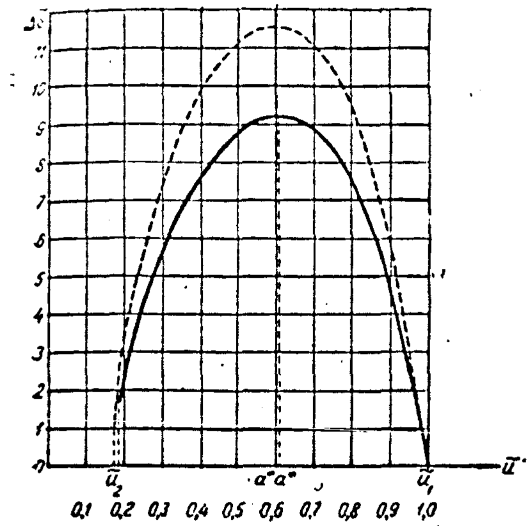

Figure 15. Change of unmeasured magnitude of entropy in dependence on unmeasured speed u inside a compression wave in monatomic hydrogen (solid curve) and in argon (dotted curve); $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

coordinate x, it is necessary to give in manifest form the coefficient of opacity α as a function of already known magnitudes. Calculation of α presents great difficulties of a physical character. We will limit ourselves, in the quality of examples of calculation, to utilization of the formula of Chandrasekar known from statistical mechanics [2]. In the case of single ionization, it is written as

$$\alpha_a = c_1 \frac{n_e}{T^{7/2}}\left(1+\frac{2{,}43}{kT}\chi_1\right) z\chi_1, \tag{12}$$

where the constant magnitude c_1 equals:

$$c_1 = \frac{40}{\pi^4 \sqrt{3}} \frac{e^2 h^4}{c m_e (2\pi m_e)^{3/2} k^{7/2}}$$

(c - speed of light in a vacuum, e - charge of the electron, α_a - coefficient of opacity in calculation for one atom). In calculation for a unit of mass we obtain

$$\alpha = c_2 \frac{\tilde{p}}{\tilde{T}^{7/2}}\left(1+\frac{c_3}{\tilde{T}}\right) z^2, \tag{13}$$

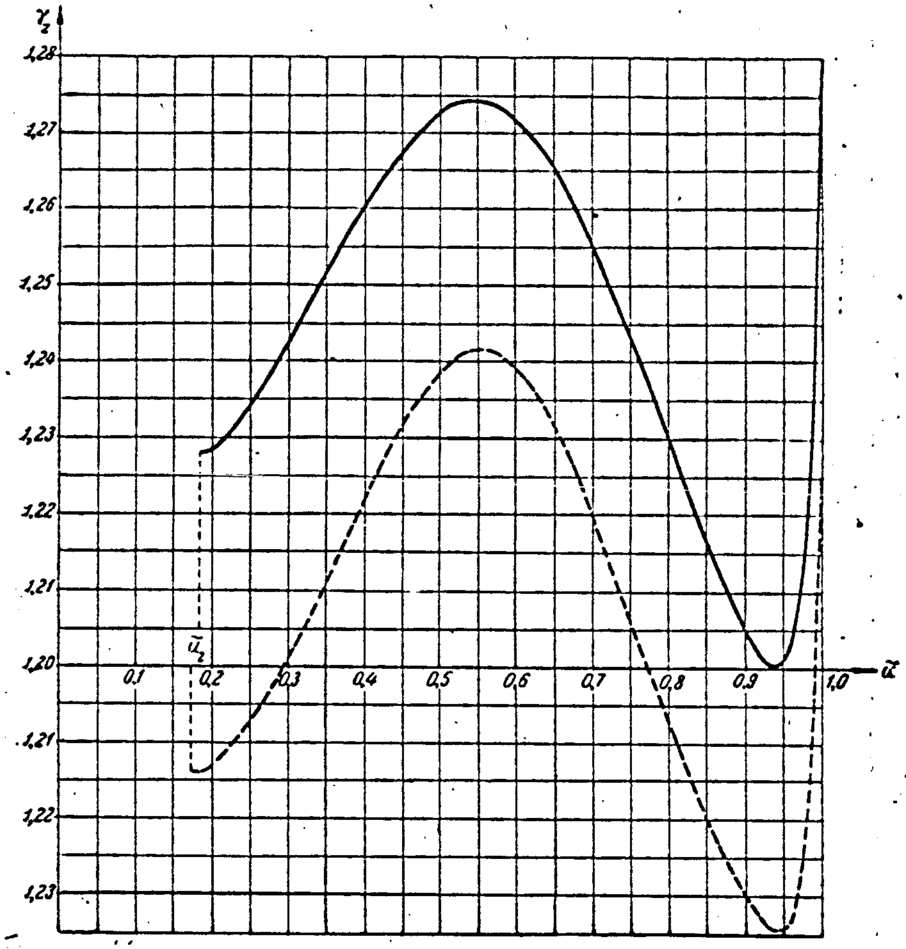

Figure 16. Relation of specific heats γ with calculation of ionization in dependence on unmeasured speed $\tilde{u}$ inside a compression wave in monatomic hydrogen (solid curve) and in argon (dotted curve); $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

M_1	3	5	8	10
z	$3{,}5 \cdot 10^{-27}$	$1{,}2 \cdot 10^{-10}$	$4 \cdot 10^{-4}$	$1{,}7 \cdot 10^{-2}$
$\left\lvert \frac{dz}{d\tilde{u}} \right\rvert_{max}$	$4{,}4 \cdot 10^{-26}$	$1{,}0 \cdot 10^{-7}$	$2{,}4 \cdot 10^{-3}$	$8{,}0 \cdot 10^{-2}$
$\left\lvert \frac{d^2z}{d\tilde{u}^2} \right\rvert_{max}$	$\sim 10^{-25}$	$\sim 10^{-8}$	$\sim 0{,}01$	~ 1

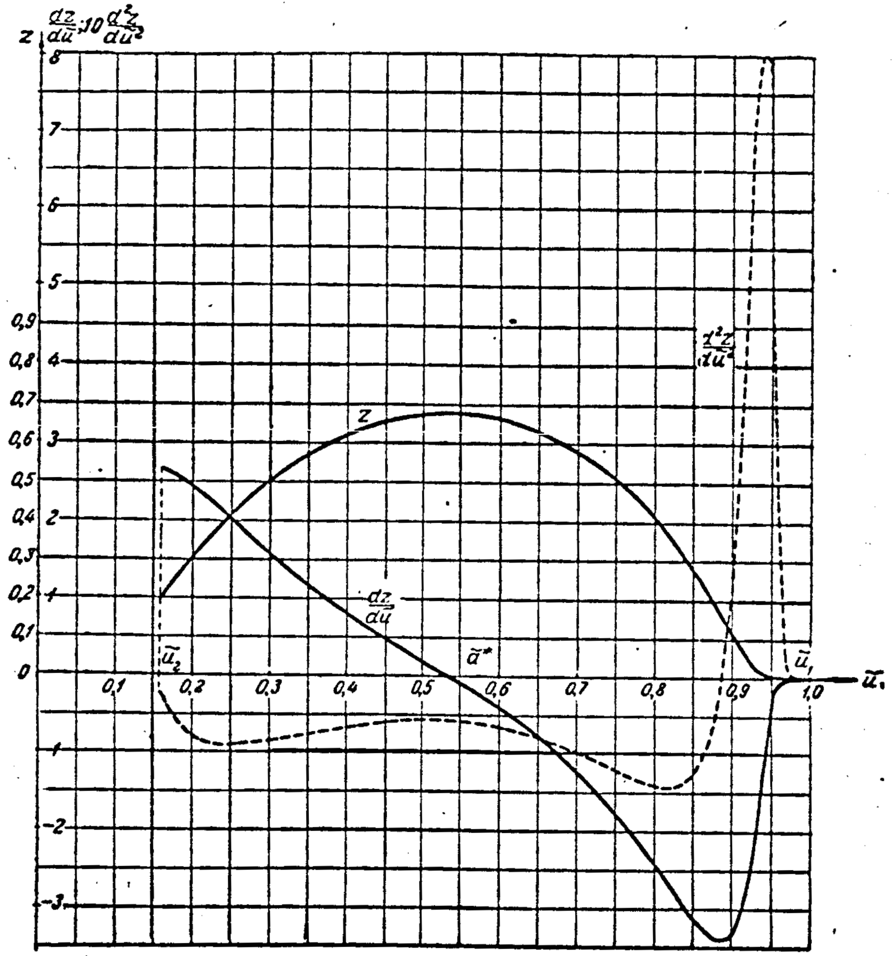

Figure 17. Degree of ionization z and derivatives $\frac{dz}{d\tilde{u}}$ and $\frac{d^2z}{d\tilde{u}^2}$ in dependence on speed $\tilde{u}$ inside a compression wave in monatomic hydrogen; $M_1 = 20$; $p_1 = 0.992 \cdot 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute.

here

$$c_2 = \frac{c_1}{m_a^2} \frac{\rho_1}{T_1^{1/2}} \chi_1;$$

$$c_3 = \frac{2{,}43}{kT_1} \chi_1.$$

Utilizing these expressions, we may evaluate by the formula:

$$x = \frac{1}{\rho_1} \int_{\tilde{a}^*}^{\tilde{u}} \frac{\tilde{u}\, d\tilde{u}}{\alpha y}, \tag{14}$$

which follows from (16, section 3) and (7, section 5), all magnitudes in dependence on x.

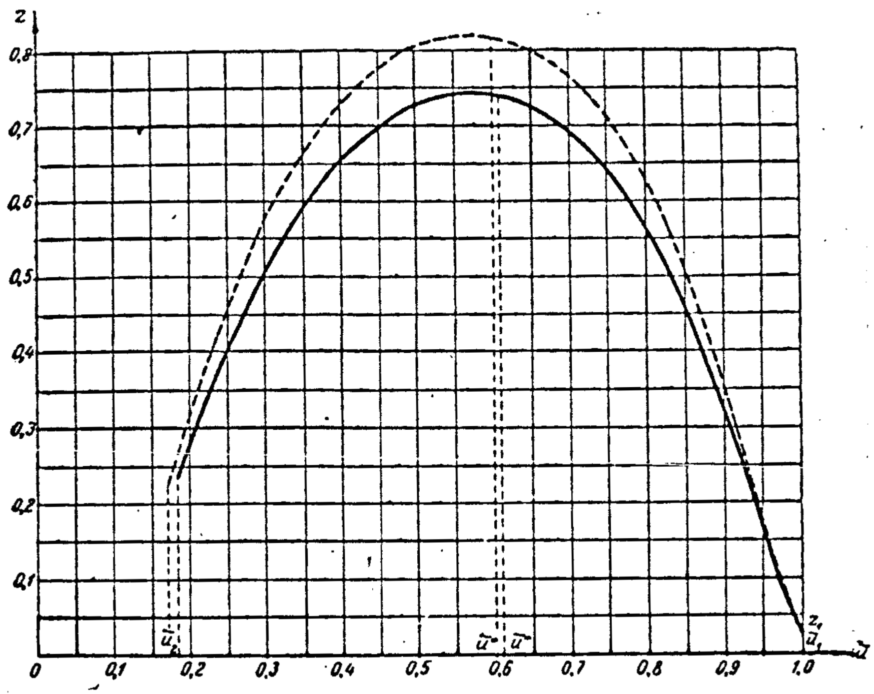

Figure 18. Degree of ionization z and derivatives $\frac{dz}{d\tilde{u}}$ and $\frac{d^2z}{d\tilde{u}^2}$ in dependence on unmeasured speed $\tilde{u}$ inside a compression wave in monatomic hydrogen (solid line) and in argon (dotted line); $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

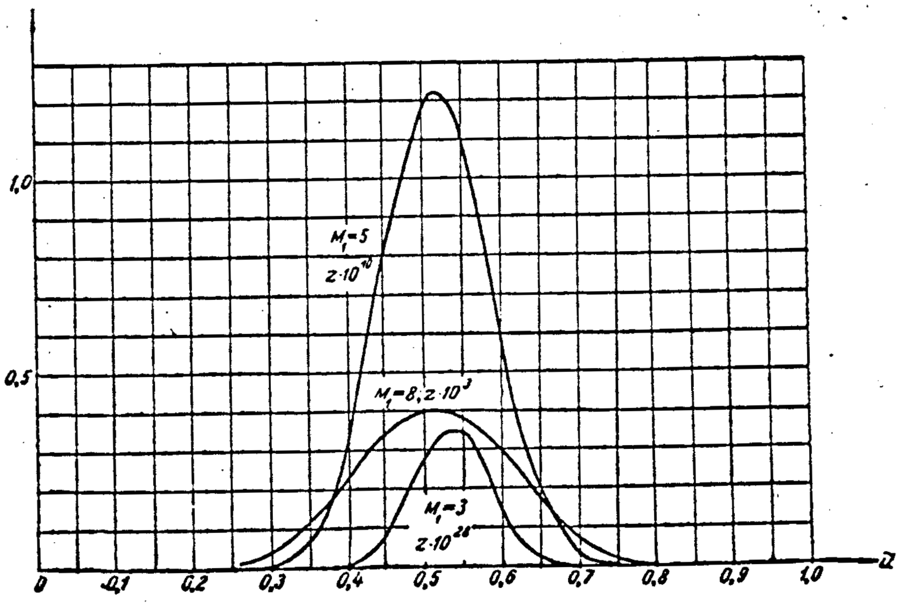

Figure 19. Degree of ionization of monatomic hydrogen inside a compression wave in dependence on $\tilde{u}$ ($p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute) with $M_1 = 3; 5; 8.$

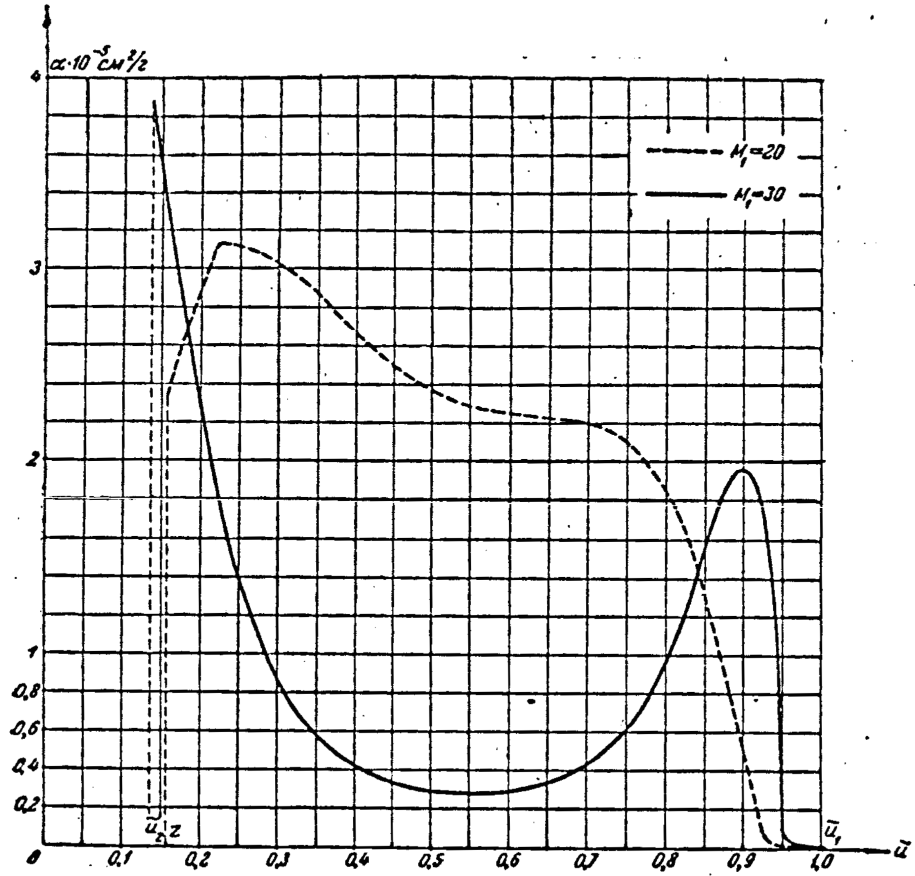

Figure 20. Coefficient of opacity of monatomic hydrogen in dependence on speed $\tilde{u}$ inside a compression wave; $M_1 = 20$ and $M_1 = 30$.

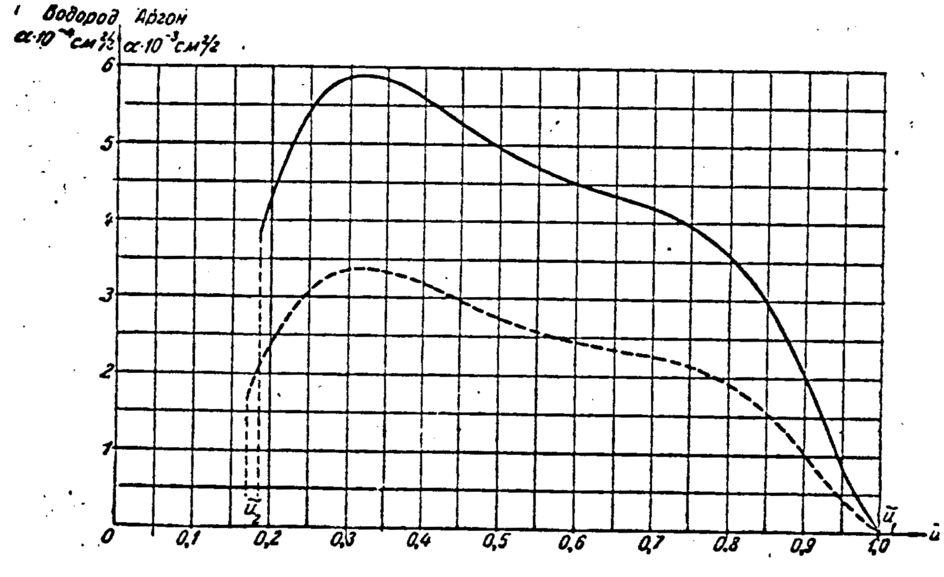

Figure 21. Coefficient of opacity α inside a compression wave in monatomic hydrogen (solid curve) and in argon (dotted curve) in dependence on unmeasured speed $\tilde{u}$; $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

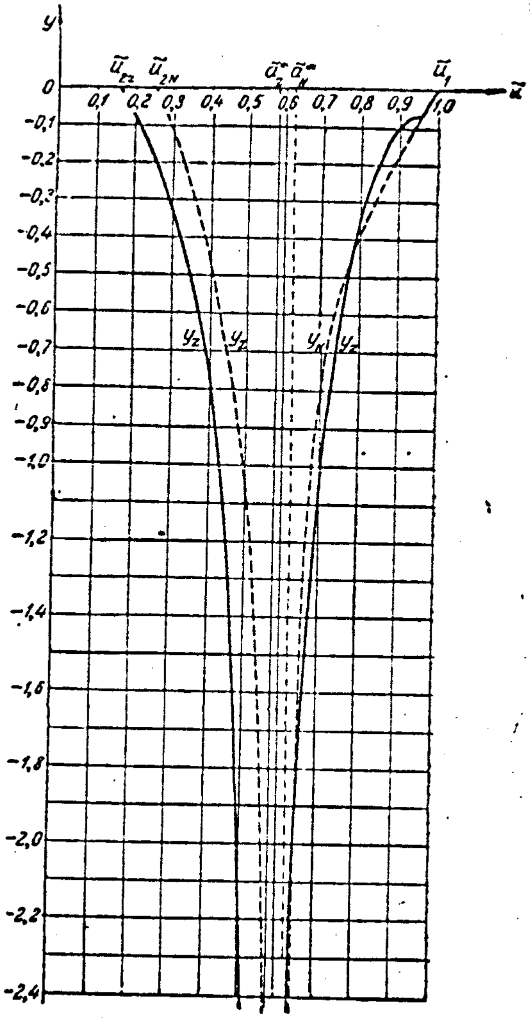

Figure 22. Integral curves of the fundamental equation (17, section 5) with calculation of ionization (index z) and for homogeneous gas (index N) with $M_1 = 20$.

If the degree of ionization is small and it may be disregarded in comparison with unity, then formula (6, section 5) may be written as:

$$\left.\begin{aligned} z &= \sqrt{\omega_1}\,\tilde{T}^{\frac{3}{4}}\tilde{u}^{\frac{1}{2}}e^{-\frac{b_1}{\tilde{T}}}; \\ \omega_1 &= \omega_0'\mu_H\frac{T_1^{\frac{3}{2}}}{\rho_1}; \quad b_1 = \frac{\chi_1}{kT_1}. \end{aligned}\right\} \qquad (15)$$

If temperatures are not very great, then in formula (13) unity as compared with the member $\frac{c_3}{T}$ may be disregarded, and we obtain:

$$\alpha = c_2 c_3 \tilde{\rho}\,\tilde{T}^{-\frac{9}{2}} z^2. \qquad (16)$$

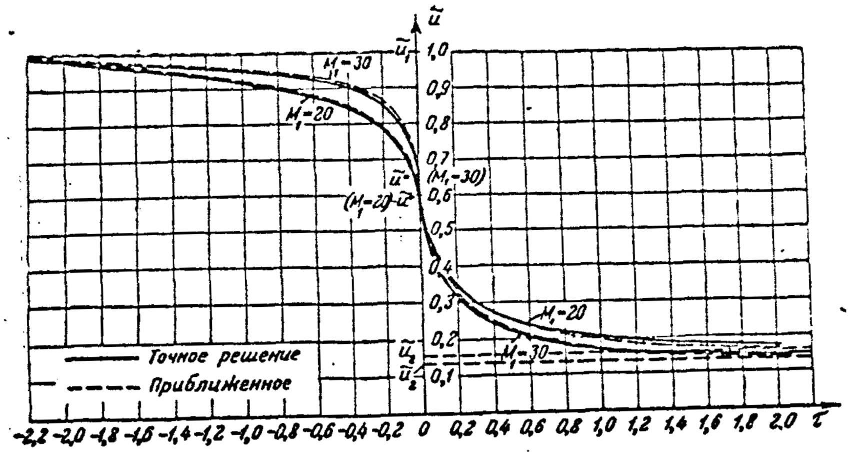

Figure 23. Unmeasured speed $\tilde{u}$ inside a compression wave in dependence on optical thickness τ (monatomic hydrogen, $p_1 = 0.992 \cdot 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute). Solid curve - precise solution, dotted - approximated by formula (9).

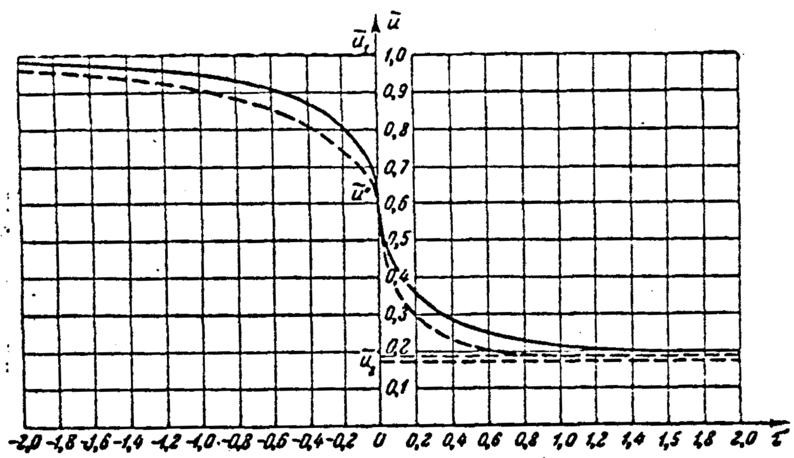

Figure 24. Unmeasured speed u inside a compression wave in monatomic hydrogen (solid line) and in argon (dotted curve) in dependence on τ ($M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute).

Substituting here for z expression (15), we find:

$$\left.\begin{aligned} \alpha &= \omega_2 T^{-3} e^{-\frac{b_1}{\tilde{T}}}; \\ \omega_2 &= c_2 c_3 \sqrt{\omega_1}. \end{aligned}\right\} \tag{17}$$

In the case of $b_2 = 0$, dependence of relative speed on the measured coordinate in force (9) and on correlations of section 5 for $\tilde{T}$ and $\tilde{\rho}$ are expressed by the quadrature:

$$x = \pm A \int_{\tilde{a}^*}^{\tilde{u}} \frac{(\tilde{u} - \beta_4)(\beta_1 - \tilde{u})^3 \tilde{u}^4}{(\tilde{u} - 1)(\tilde{u} - \beta_3)} e^{\frac{\beta_5}{[\tilde{u}(\beta_1 - \tilde{u})]}} d\tilde{u}, \qquad (18)$$

where

$$A = \frac{2\alpha_0 q_1^{\frac{3}{2}}}{\omega_0' c_2 c_n}.$$

In the maximum case, when $\rho\alpha \to 0$, the equation

$$\tau = \int_0^x \rho\alpha \, dx \qquad (19)$$

leads to the result that $\tau_1 \to 0$ in any final x, and from (9) it follows that in this case $\tilde{u} = \tilde{a}^*$, i.e., we have movement of gas with the speed of sound, the only continuous solution for the case of ideal fluid.

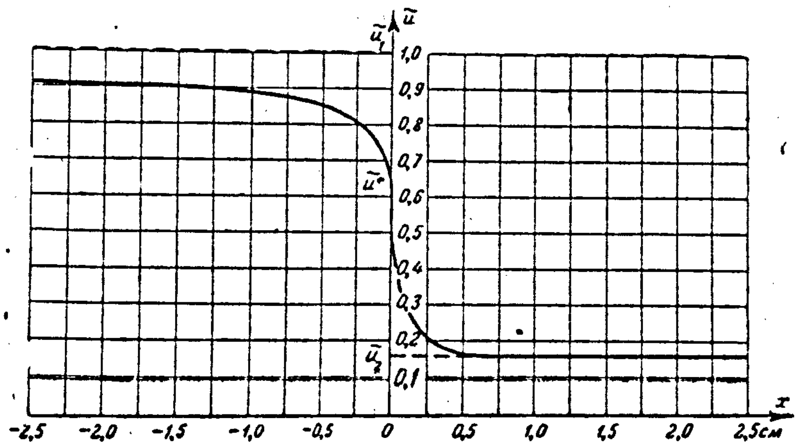

Figure 25. Unmeasured speed $\tilde{u}$ inside of a compression wave in monatomic hydrogen ($p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute) with $M_1 = 20$.

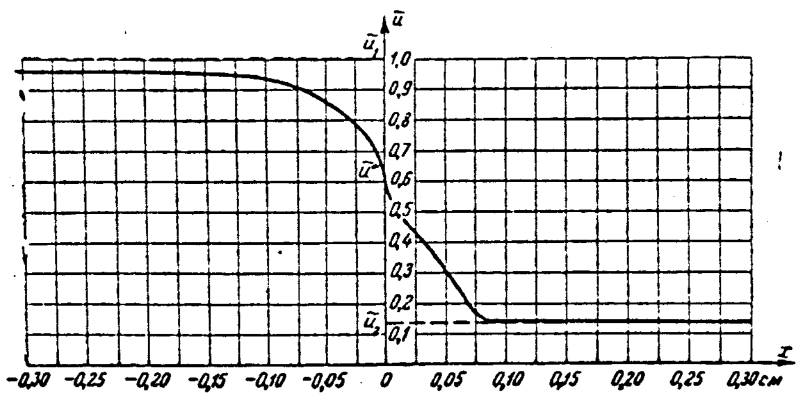

Figure 26. The same as Figure 25, but with $M_1 = 30$.

From formulas (13) and (6, section 5), it is seen that the coefficient of opacity α is small not only with a small z, in other words, with small temperature $\tilde{T}$, but also with very large $\tilde{T}$ (in the latter case $z \to 1$). However, with such conditions, formulas (6, section 5) and (13) become invalid, and new correlations are needed for calculation of z and α.

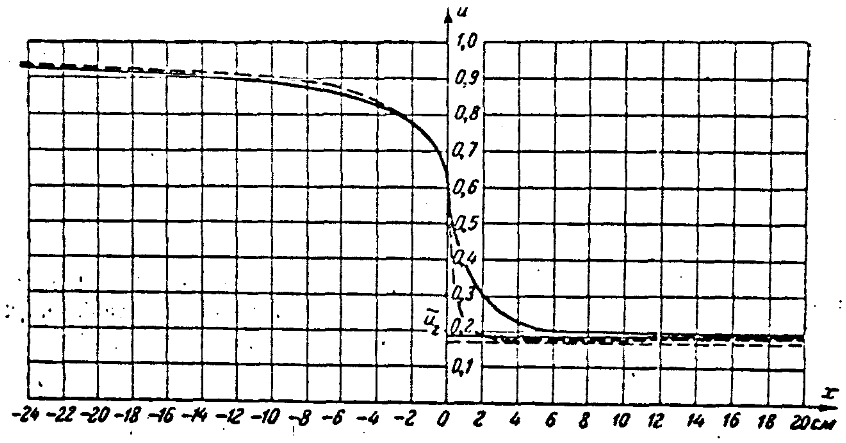

Figure 27. Unmeasured speed $\tilde{u}$ inside a compression wave in monatomic hydrogen (solid line) and in argon (dotted curve); $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

In the quality of examples of investigation of the structure of a compression wave in monatomic gas with calculation of transfer of heat due to radiation and single ionization, we will introduce results of calculations of two cases: 1) of a very intensive compression wave ($M_1 = 20$ and $M_1 = 30$) in monatomic hydrogen under conditions near to normal: $p_1 \cong 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute; 2) of a compression wave with $M_1 = 3$ in strongly heated monatomic hydrogen and argon; $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute. In all these examples $K^* > > 1$ and integral curves emerging from special points A and B stretch to infinity with $\tilde{a} \to \tilde{u}^*$ (Figure 22). Results of computation for $\alpha_0 = \frac{1}{\sqrt{3}}$ and $\alpha_0 = ½$ are presented in Figures 23 to 31, whereto in some figures are depicted for comparison also corresponding values for homogeneous

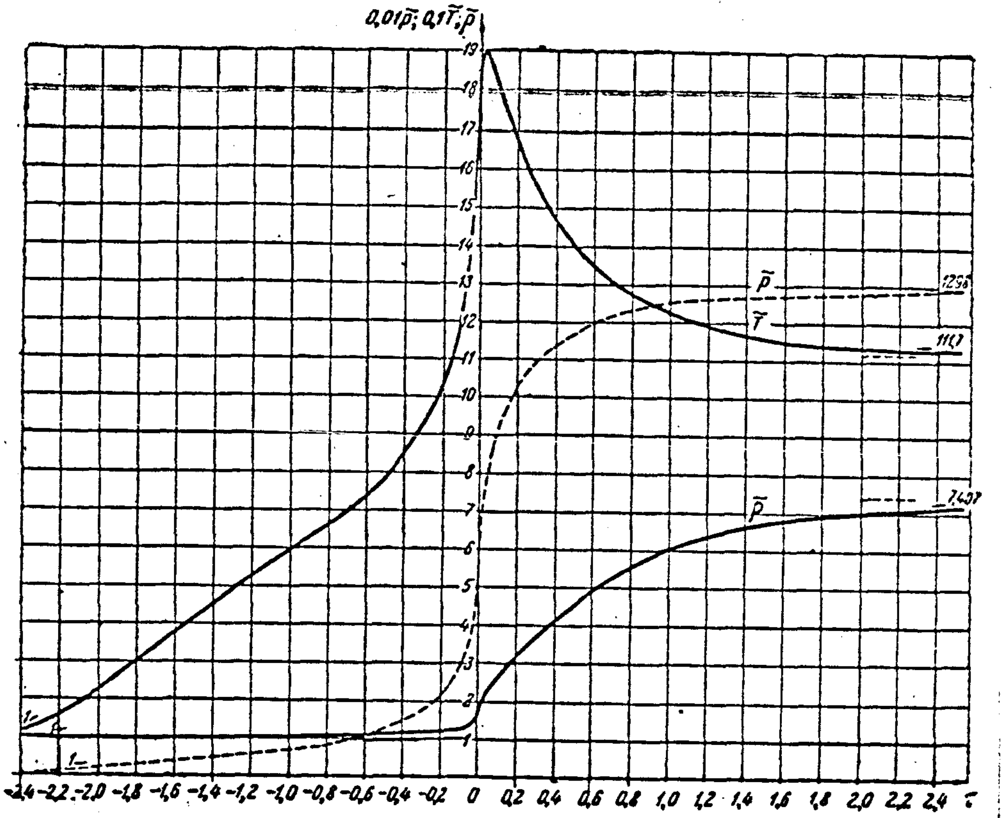

Figure 28. Relative pressure $\tilde{p}$, temperature $\tilde{T}$, and density $\tilde{\rho}$ inside a compression wave in dependence on τ; $M_1 = 30$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute.

gas (designated with the index N in distinction from the index z for ionized gas).

Thickness of a wave. The field of change of functions characterizing the flow of gas inside a compression wave ranges from $-\infty$ to $+\infty$. However, these changes take place so quickly that a quite narrow layer may be isolated outside of which functions are nearly constant. For thickness of a compression wave is ordinarly also accepted the thickness of the mentioned layer. With such a determination, of course, great arbitrariness is admitted.

We will take for the thickness of a compression wave t_τ the width of the layer where speed u changes from $\tilde{u}_{1.1} < 1$ to $\tilde{u}_{2.1} > \tilde{u}_2$, whereto

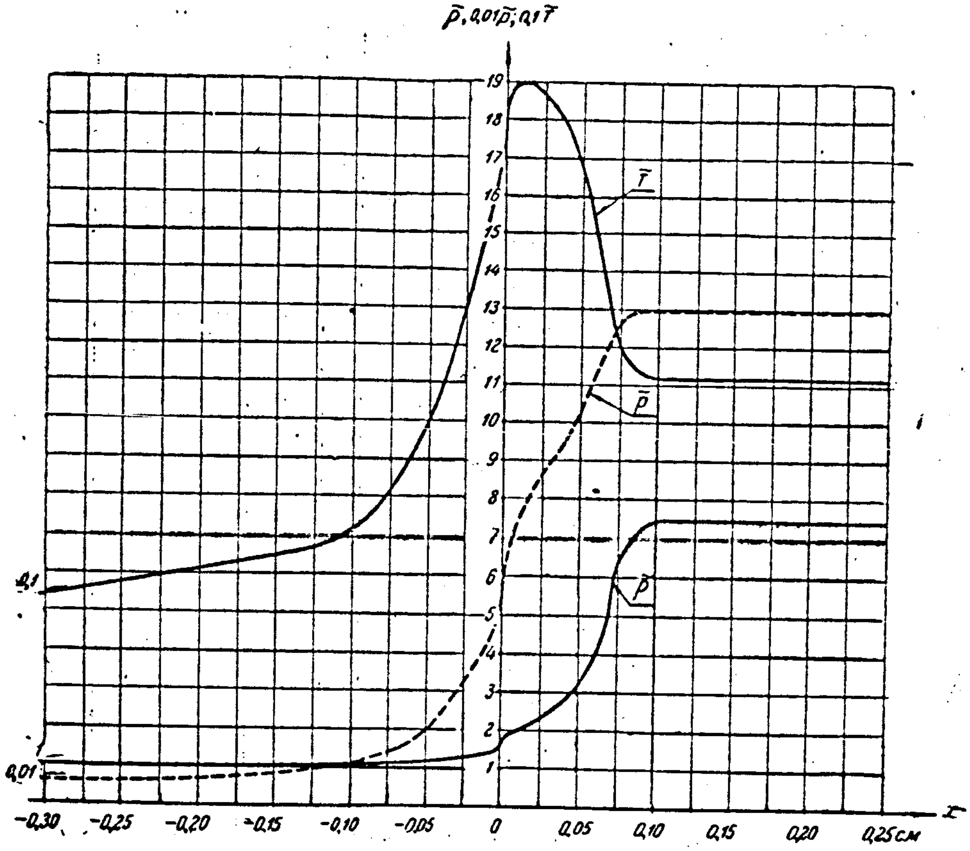

Figure 29. Relative pressure $\tilde{p}$, temperature $\tilde{T}$, and density $\tilde{\rho}$ inside of a compression wave in monatomic hydrogen; $M_1 = 30$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute.

$$\left.\begin{aligned}\tilde{u}_{1,1} &= \tilde{a}^* + \xi(1 - \tilde{a}^*);\\ \tilde{u}_{2,1} &= \tilde{a}^* - \xi(1 - \tilde{u}_2),\end{aligned}\right\} \tag{19'}$$

where ξ is a proper fraction. Thus we have:

$$t_\tau = \int\limits_{\tilde{u}_{2,1}}^{\tilde{u}_{1,1}} \frac{d\tilde{u}}{y}, \tag{20}$$

which in approximation (9) gives:

$$t_\tau = -\alpha_0 \ln \frac{a_4(\tilde{u}_{1,1})\, a_4(\tilde{u}_{2,1})}{[a_4(\tilde{a}^*)]^2}. \tag{21}$$

For homogeneous gas (z = const), the latter equality gives:

$$t_\tau = -2\alpha_0 \ln(1 - \xi^2). \tag{22}$$

Utilizing formulas (20) and (21) we obtain the values t_τ, introduced in Figure 33 (monatomic hydrogen, $p_1 \approx 10^6 \frac{\text{dynes}}{\text{cm}^2}$; $T_1 = 300$ degrees absolute,

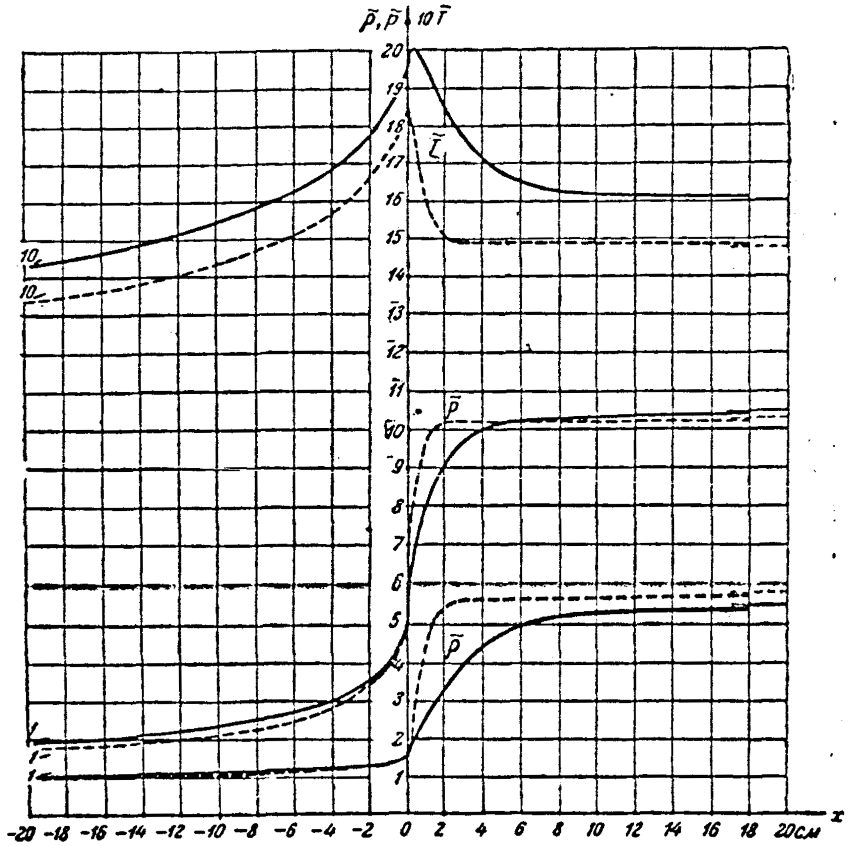

Figure 30. Relative pressures $\tilde{p}$, temperature $\tilde{T}$, and density $\tilde{\rho}$ inside a compression wave in monatomic hydrogen (solid curves) and in argon (dotted lines); $M_1 = 3$, $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute.

$\xi = 0.8$), and namely: $t_\tau = 1.56$ ($M_1 = 20$) and $t_\tau = 1.04$ ($M_1 = 30$); for $\xi = 0.9$ the corresponding values will be 2.75 and 2.03.

In the case of diffusion of waves in monatomic hydrogen and argon with $M_1 = 3$ ($p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute, $\xi = 018$) we obtain: $t_\tau = 1.21$ (hydrogen) and $t_\tau = 1.46$ (argon). If we take $\xi = 0.9$, then in the latter case we have: $t_\tau = 2.05$ (hydrogen) and 2.44 (argon). Optical thickness may be considered the length measured in mean lengths of a free run of radiation [6], if the act of radiation of a light quantum is considered the beginning and the act of absorption the end of the free path of the run of radiation. "Optical thickness" of a wave equals from 1 to 2, i.e., the order of length of a free run of radiation.

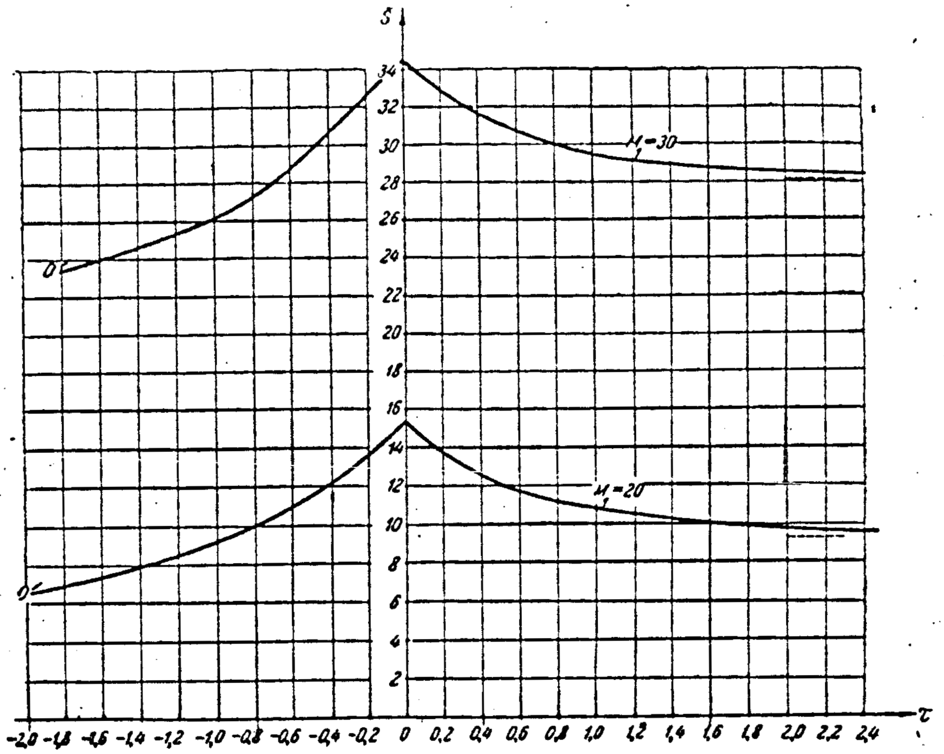

Figure 31. Change of unmeasured value of entropy $\Delta\tilde{s} = \tilde{s}(\tilde{u}) - \tilde{s}(1)$ inside a compression wave in monatomic hydrogen in dependence on optical thickness τ for $M_1 = 20$ and $M_1 = 30$.

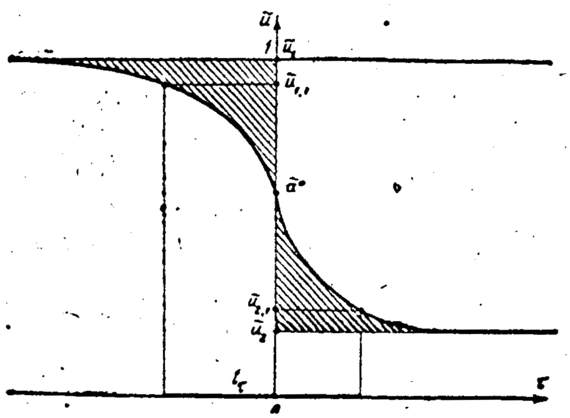

Figure 32. Determination of thickness of a compression wave.

Analogically, the measured thickness of a compression wave may also be determined. Utilizing formula (13) for determination of the coefficient of opacity α, we obtain for monatomic hydrogen ($p_1 \cong 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute, $\xi = 0.8$):

$$t_x = 3.17 \text{ cm } (M_1 = 20) \text{ and}$$

$$t_x = 0.16 \text{ cm } (M_1 = 30).$$

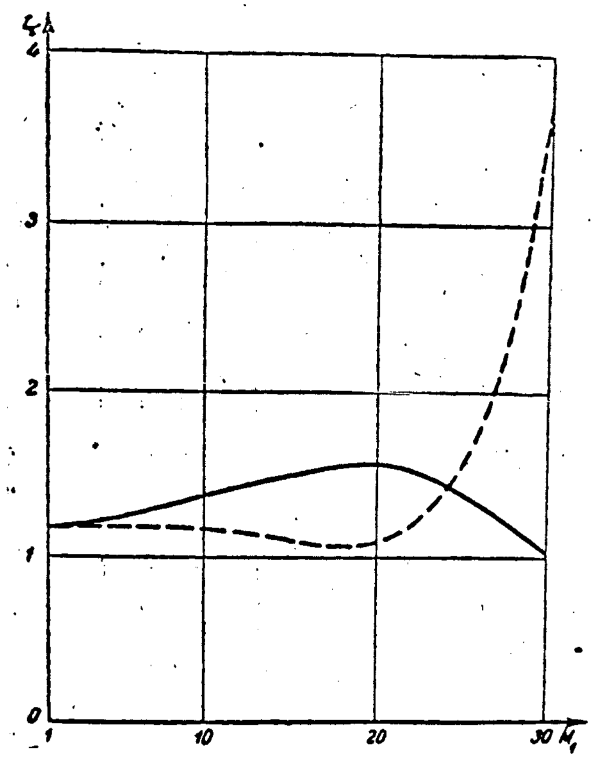

Figure 33. Optical thickness of a compression wave in monatomic hydrogen with $p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 300$ degrees absolute with calculation of ionization (solid curve) and in homogeneous gas (dotted line).

Further increase of the number M_1 again leads to increase of t_x because of the decrease of the coefficient of opacity. For other conditions ($p_1 = 10^6 \frac{\text{dynes}}{\text{cm}^2}$, $T_1 = 10^4$ degrees absolute, $M_1 = 3$, $\xi = 0.8$) we have: $t_x = 23.4$ cm (monatomic hydrogen) and $t_x = 16.0$ cm (argon).

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